

# **Differential Equations and Linear Algebra**

## **Lecture Notes**

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## Linear second order ODEs

### 1.1. Newton's second law

We shall begin by stating Newton's fundamental kinematic law relating the force, mass and acceleration of an object whose position is  $y(t)$  at time  $t$ .

*Newton's second law* states that the force  $F$  applied to an object is equal to its mass  $m$  times its acceleration  $\frac{d^2y}{dt^2}$ , i.e.

$$F = m \frac{d^2y}{dt^2}.$$

**1.1.1. Example.** Find the position/height  $y(t)$ , at time  $t$ , of a body falling freely under gravity (take the convention, that we measure positive displacements upwards).

**1.1.2. Solution.** The equation of motion of a body falling freely under gravity, is, by Newton's second law,

$$\frac{d^2y}{dt^2} = -g. \quad (1.1)$$

We can solve equation (1.1) by integrating with respect to  $t$ , which yields an expression for the velocity of the body,

$$\frac{dy}{dt} = -gt + v_0,$$

where  $v_0$  is the constant of integration which here also happens to be the initial velocity. Integrating again with respect to  $t$  gives

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0,$$

where  $y_0$  is the second constant of integration which also happens to be the initial height of the body.

Equation (1.1) is an example of a second order differential equation (because the highest derivative that appears in the equation is second order):

- the solutions of the equation are a *family of functions with two parameters* (in this case  $v_0$  and  $y_0$ );
- choosing values for the two parameters, corresponds to choosing a particular function of the family.

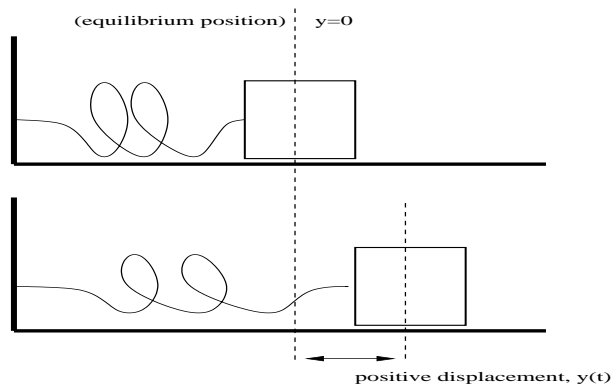


FIGURE 1.1. Mass  $m$  slides freely on the horizontal surface, and is attached to a spring, which is fixed to a vertical wall at the other end. We take the convention that positive displacements are measured to the right.

## 1.2. Springs and Hooke's Law

Consider a mass  $m$  Kg on the end of a spring, as in Figure 1.1. With the initial condition that the mass is pulled to one side and then released, what do you expect to happen?

*Hooke's law* implies that, provided  $y$  is not so large as to deform the spring, then the restoring force is

$$F_{\text{spring}} = -ky,$$

where the constant  $k > 0$  depends on the properties of the spring, for example its stiffness.

**1.2.1. Equation of motion.** Combining Hooke's law and Newton's second law implies

$$\begin{aligned} & m \frac{d^2 y}{dt^2} = -ky \\ \text{(assuming } m \neq 0) & \quad \Leftrightarrow \quad \frac{d^2 y}{dt^2} = -\frac{k}{m}y \\ \text{(setting } \omega = +\sqrt{k/m}) & \quad \Leftrightarrow \quad \frac{d^2 y}{dt^2} = -\omega^2 y \\ & \quad \Leftrightarrow \quad \frac{d^2 y}{dt^2} + \omega^2 y = 0. \end{aligned} \tag{1.2}$$

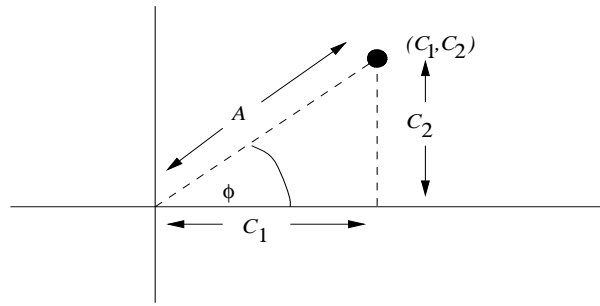
Can we guess a solution of (1.2), i.e. a function that satisfies the relation (1.2)? When written in the form on the line just above (1.2), we see that we are essentially asking ourselves: what function, when you differentiate it twice, gives you minus  $\omega^2$  times the original function you started with?

The *general solution* to (1.2) is

$$y(t) = C_1 \sin \omega t + C_2 \cos \omega t, \tag{1.3}$$

where  $C_1$  and  $C_2$  are arbitrary constants. This is an oscillatory solution with *frequency of oscillation*  $\omega$ . The *period* of the oscillations is

$$T = \frac{2\pi}{\omega}.$$

FIGURE 1.2. Relation between  $(C_1, C_2)$  and  $(A, \phi)$ .

Recall that we set  $\omega = +\sqrt{k/m}$  and this parameter represents the frequency of oscillations of the mass. How does the general solution change as you vary  $m$  and  $k$ ? Does this match your physical intuition? *What do these solutions really look like?*

We can re-express the solution (1.3) as follows. Consider the  $C_1, C_2$  plane as shown in Figure 1.2. Hence

$$\begin{aligned} C_1 &= A \cos \phi, \\ C_2 &= A \sin \phi. \end{aligned}$$

Substituting these expressions for  $C_1$  and  $C_2$  into (1.3) we get

$$\begin{aligned} y(t) &= A \cos \phi \sin \omega t + A \sin \phi \cos \omega t \\ &= A(\cos \phi \sin \omega t + \sin \phi \cos \omega t) \\ &= A \sin(\omega t + \phi). \end{aligned}$$

The general solution (1.2) can *also* be expressed in the form

$$y(t) = A \sin(\omega t + \phi), \quad (1.4)$$

where

$$\begin{aligned} A &= +\sqrt{C_1^2 + C_2^2} \geq 0 \text{ is the } \textit{amplitude} \text{ of oscillation,} \\ \phi &= \arctan(C_2/C_1) \text{ is the } \textit{phase angle}, \text{ with } -\pi < \phi \leq \pi. \end{aligned}$$

**1.2.2. Example (initial value problem).** Solve the differential equation for the spring,

$$\frac{d^2 y}{dt^2} = -\frac{k}{m} y,$$

if the mass were displaced by a distance  $y_0$  and then released. This is an example of an *initial value problem*, where the initial position and the initial velocity are used to determine the solution.

**1.2.3. Solution.** We have already seen that the position of the mass at time  $t$  is given by

$$y(t) = C_1 \sin \omega t + C_2 \cos \omega t, \quad (1.5)$$

with  $\omega = +\sqrt{k/m}$ , for some constants  $C_1$  and  $C_2$ . The initial position is  $y_0$ , i.e.  $y(0) = y_0$ . Substituting this information into (1.5), we see that

$$\begin{aligned} y(0) &= C_1 \sin(\omega \cdot 0) + C_2 \cos(\omega \cdot 0) \\ \Leftrightarrow y_0 &= C_1 \cdot 0 + C_2 \cdot 1 \\ \Leftrightarrow y_0 &= C_2. \end{aligned}$$

The initial velocity is zero, i.e.  $y'(0) = 0$ . Differentiating (1.5) and substituting this information into the resulting expression for  $y'(t)$  implies

$$\begin{aligned} y'(0) &= C_1 \omega \cos(\omega \cdot 0) - C_2 \omega \sin(\omega \cdot 0) \\ \Leftrightarrow 0 &= C_1 \omega \cdot 1 - C_2 \omega \cdot 0 \\ \Leftrightarrow 0 &= C_1. \end{aligned}$$

Therefore the solution is  $y(t) = y_0 \cos \omega t$ . Of course this is an oscillatory solution with frequency of oscillation  $\omega$ , and in this case, the amplitude of oscillation  $y_0$ .

**1.2.4. Damped oscillations.** Consider a more realistic spring which has *friction*.

In general, the *frictional force* or *drag* is proportional to the velocity of the mass, i.e.

$$F_{\text{friction}} = -C \frac{dy}{dt},$$

where  $C$  is a constant known as the *drag* or *friction* coefficient. The frictional force acts in a direction opposite to that of the motion and so  $C > 0$ .

Newton's Second Law implies (adding the restoring and frictional forces together)

$$m \frac{d^2 y}{dt^2} = F_{\text{spring}} + F_{\text{friction}},$$

i.e.

$$m \frac{d^2 y}{dt^2} = -ky - C \frac{dy}{dt}.$$

Hence the damped oscillations of a spring are described by the differential equation

$$m \frac{d^2 y}{dt^2} + C \frac{dy}{dt} + ky = 0. \quad (1.6)$$

**1.2.5. Remark.** We infer the general principles: for elastic solids, stress is proportional to strain (how far you are pulling neighbouring particles apart), whereas for fluids, stress is proportional to the rate of strain (how fast you are pulling neighbouring particles apart). Such fluids are said to be Newtonian fluids, and everyday examples include water and simple oils etc. There are also many non-Newtonian fluids. Some of these retain some solid-like elasticity properties. Examples include solutes of long-chain protein molecules such as saliva.

### 1.3. General ODEs and their classification

**1.3.1. Basic definitions.** The basic notions of differential equations and their solutions can be outlined as follows.

- *Differential Equation (DE)*. An equation relating two or more variables in terms of derivatives or differentials.
- *Solution of a Differential Equation*. Any functional relation, *not* involving derivatives or integrals of “unknown” functions, which satisfies the differential equation.
- *General Solution*. A description of all the functional relations that satisfy the differential equation.
- *Ordinary Differential Equation (ODE)*. A differential equation relating only two variables. A general  $n^{\text{th}}$  order ODE is often represented by

$$F\left(t, y, \frac{dy}{dt}, \dots, \frac{d^n y}{dt^n}\right) = 0, \quad (1.7)$$

where  $F$  is some given (known) function.

- In equation (1.7), we usually call  $t$  the *independent variable*, and  $y$  is the *dependent variable*.

**1.3.2. Example.** Newton’s second law implies that, if  $y(t)$  is the position, at time  $t$ , of a particle of mass  $m$  acted upon by a force  $f$ , then

$$\frac{d^2 y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right),$$

where the given force  $f$  may be a function of  $t$ ,  $y$  and the velocity  $\frac{dy}{dt}$ .

**1.3.3. Classification of ODEs.** ODEs are classified according to *order*, *linearity* and *homogeneity*.

- *Order*. The *order* of a differential equation is the order of the highest derivative present in the equation.
- *Linear or nonlinear*. A second order ODE is said to be *linear* if it can be written in the form

$$a(t)\frac{d^2 y}{dt^2} + b(t)\frac{dy}{dt} + c(t)y = f(t), \quad (1.8)$$

where the *coefficients*  $a(t)$ ,  $b(t)$  &  $c(t)$  can, in general, be functions of  $t$ . An equation that is *not* linear is said to be *nonlinear*. Note that linear ODEs are characterised by two properties:

- (1) The dependent variable and all its derivatives are of *first degree*, i.e. the power of each term involving  $y$  is 1.
  - (2) Each coefficient depends on the independent variable  $t$  *only*.
- *Homogeneous or non-homogeneous*. The linear differential equation (1.8) is said to be *homogeneous* if  $f(t) \equiv 0$ , otherwise, if  $f(t) \neq 0$ , the differential equation is said to be *non-homogeneous*. More generally, an equation is said to be *homogeneous* if  $ky(t)$  is a solution whenever  $y(t)$  is also a solution, for any constant  $k$ , i.e. the equation is invariant under the transformation  $y(t) \rightarrow ky(t)$ .

**1.3.4. Example.** The differential equation

$$\frac{d^2y}{dt^2} + 5 \left( \frac{dy}{dt} \right)^3 - 4y = e^t,$$

is second order because the highest derivative is second order, and nonlinear because the second term on the left-hand side is cubic in  $y'$ .

**1.3.5. Example (higher order linear ODEs).** We can generalize our characterization of a *linear* second order ODE to higher order linear ODEs. We recognize that a linear third order ODE must have the form

$$a_3(t) \frac{d^3y}{dt^3} + a_2(t) \frac{d^2y}{dt^2} + a_1(t) \frac{dy}{dt} + a_0(t)y = f(t), \quad (1.9)$$

for a given set of coefficient functions  $a_3(t)$ ,  $a_2(t)$ ,  $a_1(t)$  and  $a_0(t)$ , and a given inhomogeneity  $f(t)$ . A linear fourth order ODE must have the form

$$a_4(t) \frac{d^4y}{dt^4} + a_3(t) \frac{d^3y}{dt^3} + a_2(t) \frac{d^2y}{dt^2} + a_1(t) \frac{dy}{dt} + a_0(t)y = f(t), \quad (1.10)$$

while a general  $n^{\text{th}}$  order linear ODE must have the form

$$a_n(t) \frac{d^n y}{dt^n} + a_{n-1}(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_2(t) \frac{d^2 y}{dt^2} + a_1(t) \frac{dy}{dt} + a_0(t)y = f(t). \quad (1.11)$$

**1.3.6. Example (scalar  $n^{\text{th}}$  order ODE as a system of  $n$  first order ODEs).** Any  $n^{\text{th}}$  order ODE (linear or nonlinear) can always be written as a system of  $n$  first order ODEs. For example, if for the ODE

$$F \left( t, y, \frac{dy}{dt}, \dots, \frac{d^n y}{dt^n} \right) = 0, \quad (1.12)$$

we identify new variables for the derivative terms of each order, then (1.12) is equivalent to the system of  $n$  first order ODEs

$$\frac{dy}{dt} = u_1, \quad \frac{du_1}{dt} = u_2, \quad \dots, \quad \frac{du_{n-2}}{dt} = u_{n-1}, \quad F \left( t, y, u_1, u_2, \dots, u_{n-1}, \frac{du_{n-1}}{dt} \right) = 0.$$

## Homogeneous linear ODEs

### 2.1. The Principle of Superposition

**2.1.1. Principle of Superposition for linear homogeneous differential equations.** Consider the linear, second order, homogeneous, ordinary differential equation

$$a(t)\frac{d^2y}{dt^2} + b(t)\frac{dy}{dt} + c(t)y = 0, \quad (2.1)$$

where  $a(t)$ ,  $b(t)$  and  $c(t)$  are known functions.

- If  $y_1(t)$  and  $y_2(t)$  satisfy (2.1), then for any two constants  $C_1$  and  $C_2$ , the *Principle of Superposition* says that

$$y(t) = C_1y_1(t) + C_2y_2(t) \quad (2.2)$$

is a solution also.

- If  $y_1(t)$  is not a constant multiple of  $y_2(t)$ , then the *general solution* of (2.1) takes the form (2.2).

### 2.2. Solving linear second order constant coefficient homogeneous ODEs

**2.2.1. Exponential solutions.** We restrict ourselves here to the case when the coefficients  $a$ ,  $b$  and  $c$  in (2.1) are constants, i.e.

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0, \quad (2.3)$$

Let us try to find a solution to (2.3) of the form

$$y = e^{\lambda t}. \quad (2.4)$$

The reason for choosing the exponential function is that we know that solutions to linear first order constant coefficient ODEs always have this form for a specific value of  $\lambda$  that depends on the coefficients. So we'll try to look for a solution to a linear second order constant coefficient ODE of the same form, where at the moment we will not specify what  $\lambda$  is—with hindsight we will see that this is a good choice.

Substituting (2.4) into (2.3) implies

$$\begin{aligned} a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy &= a\lambda^2e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} \\ &= e^{\lambda t}(a\lambda^2 + b\lambda + c) \end{aligned}$$

which must  $= 0$ .

Since the exponential function is never zero, i.e.  $e^{\lambda t} \neq 0$ , then we see that if  $\lambda$  satisfies the *auxiliary equation*:

$$a\lambda^2 + b\lambda + c = 0,$$

then (2.4) will be a solution of (2.3). There are three cases we need to consider.

**2.2.2. Case I:**  $b^2 - 4ac > 0$ . There are two real and distinct solutions to the auxiliary equation,

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

and so two functions,

$$e^{\lambda_1 t} \quad \text{and} \quad e^{\lambda_2 t},$$

satisfy the ordinary differential equation (2.3). The Principle of Superposition implies that the general solution is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

**2.2.3. Example:**  $b^2 - 4ac > 0$ . Find the general solution to the ODE

$$y'' + 4y' - 5y = 0.$$

**2.2.4. Case II:**  $b^2 - 4ac = 0$ . In this case there is one real repeated root to the auxiliary equation, namely

$$\lambda_1 = \lambda_2 = -\frac{b}{2a}.$$

Hence we have one solution, which is

$$y(t) = e^{\lambda_1 t} = e^{-\frac{b}{2a}t}.$$

However, we should suspect that there is another independent solution. It's not obvious what that might be, but let's make the educated guess

$$y = te^{\lambda_1 t}$$

where  $\lambda_1$  is the same as above, i.e.  $\lambda_1 = -\frac{b}{2a}$ . Substituting this guess for the second solution into our second order differential equation,

$$\begin{aligned} \Rightarrow a \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + cy &= a (\lambda_1^2 t e^{\lambda_1 t} + 2\lambda_1 e^{\lambda_1 t}) \\ &\quad + b (e^{\lambda_1 t} + \lambda_1 t e^{\lambda_1 t}) \\ &\quad + c t e^{\lambda_1 t} \\ &= e^{\lambda_1 t} (t (a\lambda_1^2 + b\lambda_1 + c) + (2a\lambda_1 + b)) \end{aligned}$$

$$\text{which in fact} \quad = 0,$$

since we note that  $a\lambda_1^2 + b\lambda_1 + c = 0$  and  $2a\lambda_1 + b = 0$  because  $\lambda_1 = -b/2a$ . Thus  $te^{-\frac{b}{2a}t}$  is another solution (which is clearly not a constant multiple of the first solution). The Principle of Superposition implies that the general solution is

$$y = (C_1 + C_2 t) e^{-\frac{b}{2a}t}.$$

**2.2.5. Example:**  $b^2 - 4ac = 0$ . Find the general solution to the ODE

$$y'' + 4y' + 4y = 0.$$

**2.2.6. Case III:**  $b^2 - 4ac < 0$ . In this case, there are two complex roots to the auxiliary equation, namely

$$\begin{aligned} \lambda_1 &= p + iq, \\ \lambda_2 &= p - iq, \end{aligned} \quad \text{where} \quad p = -\frac{b}{2a} \quad \text{and} \quad q = \frac{+\sqrt{|b^2 - 4ac|}}{2a}. \quad (2.5)$$

Hence the Principle of Superposition implies that the general solution takes the form

$$\begin{aligned} y(t) &= A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} \\ &= A_1 e^{(p+iq)t} + A_2 e^{(p-iq)t} \\ &= A_1 e^{pt+iqt} + A_2 e^{pt-iqt} \\ &= A_1 e^{pt} e^{iqt} + A_2 e^{pt} e^{-iqt} \\ &= e^{pt} (A_1 e^{iqt} + A_2 e^{-iqt}) \\ \text{(using Euler's formula)} \quad &= e^{pt} (A_1 (\cos qt + i \sin qt) + A_2 (\cos qt - i \sin qt)) \\ &= e^{pt} ((A_1 + A_2) \cos qt + i(A_1 - A_2) \sin qt), \end{aligned} \quad (2.6)$$

where

(1) we have used Euler's formula

$$e^{iz} \equiv \cos z + i \sin z,$$

first with  $z = qt$  and then secondly with  $z = -qt$ , i.e. we have used that

$$e^{iqt} \equiv \cos qt + i \sin qt \quad (2.7a)$$

and

$$e^{-iqt} \equiv \cos qt - i \sin qt \quad (2.7b)$$

since  $\cos(-qt) \equiv \cos qt$  and  $\sin(-qt) \equiv -\sin qt$ ;

(2) and  $A_1$  and  $A_2$  are arbitrary (and in general complex) constants—at this stage that means we appear to have a total of four constants because  $A_1$  and  $A_2$  both have real and imaginary parts. However we expect the solution  $y(t)$  to be real—the coefficients are real and we will pose real initial data.

The solution  $y(t)$  in (2.6) will be real if and only if

$$\begin{aligned} A_1 + A_2 &= C_1, \\ i(A_1 - A_2) &= C_2, \end{aligned}$$

where  $C_1$  and  $C_2$  are *real* constants. Hence the *general solution* in this case has the form

$$y(t) = e^{pt} (C_1 \cos qt + C_2 \sin qt).$$

**2.2.7. Example:**  $b^2 - 4ac < 0$ . Find the general solution to the ODE

$$2y'' + 2y' + y = 0.$$

| Case            | Roots of auxiliary equation   | General solution                                |
|-----------------|---|---|
| $b^2 - 4ac > 0$ | $\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$                                  | $y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$ |
| $b^2 - 4ac = 0$ | $\lambda_{1,2} = -\frac{b}{2a}$   | $y = (C_1 + C_2 t) e^{\lambda_1 t}$             |
| $b^2 - 4ac < 0$ | $\lambda_{1,2} = p \pm iq$<br>$p = -\frac{b}{2a}, q = \frac{+\sqrt{ b^2 - 4ac }}{2a}$ | $y = e^{pt} (C_1 \cos qt + C_2 \sin qt)$        |

TABLE 2.1. Solutions to the linear second order, constant coefficient, homogeneous ODE  $ay'' + by' + cy = 0$ .

### 2.3. Practical example: damped springs

**2.3.1. Parameters.** For the case of the damped spring note that in terms of the physical parameters  $a = m > 0$ ,  $b = C > 0$  and  $c = k > 0$ . Hence

$$b^2 - 4ac = C^2 - 4mk.$$

**2.3.2. Overdamping:**  $C^2 - 4mk > 0$ . Since the physical parameters  $m$ ,  $k$  and  $C$  are all positive, we have that

$$\sqrt{C^2 - 4mk} < |C|,$$

and so  $\lambda_1$  and  $\lambda_2$  are both negative. Thus for large times ( $t \rightarrow +\infty$ ) the solution  $y(t) \rightarrow 0$  exponentially fast. For example, the mass might be immersed in thick oil. Two possible solutions, starting from two different initial conditions, are shown in Figure 2.1(a). Whatever initial conditions you choose, there is at most one oscillation. At some point, for example past the vertical dotted line on the right, for all practical purposes the spring is in the equilibrium position.

**2.3.3. Critical damping:**  $C^2 - 4mk = 0$ . In appearance—see Figure 2.1(b)—the solutions for the critically damped case look very much like those in Figure 2.1(a) for the overdamped case.

**2.3.4. Underdamping:**  $C^2 - 4mk < 0$ . Since for the spring

$$p = -\frac{b}{2a} = -\frac{C}{2m} < 0,$$

the mass will oscillate about the equilibrium position with the amplitude of the oscillations decaying exponentially in time; in fact the solution oscillates between the *exponential envelopes* which are the two dashed curves  $Ae^{pt}$  and  $-Ae^{pt}$ , where  $A = +\sqrt{C_1^2 + C_2^2}$ —see Figure 2.1(c). In this case, for example, the mass might be immersed in light oil or air.

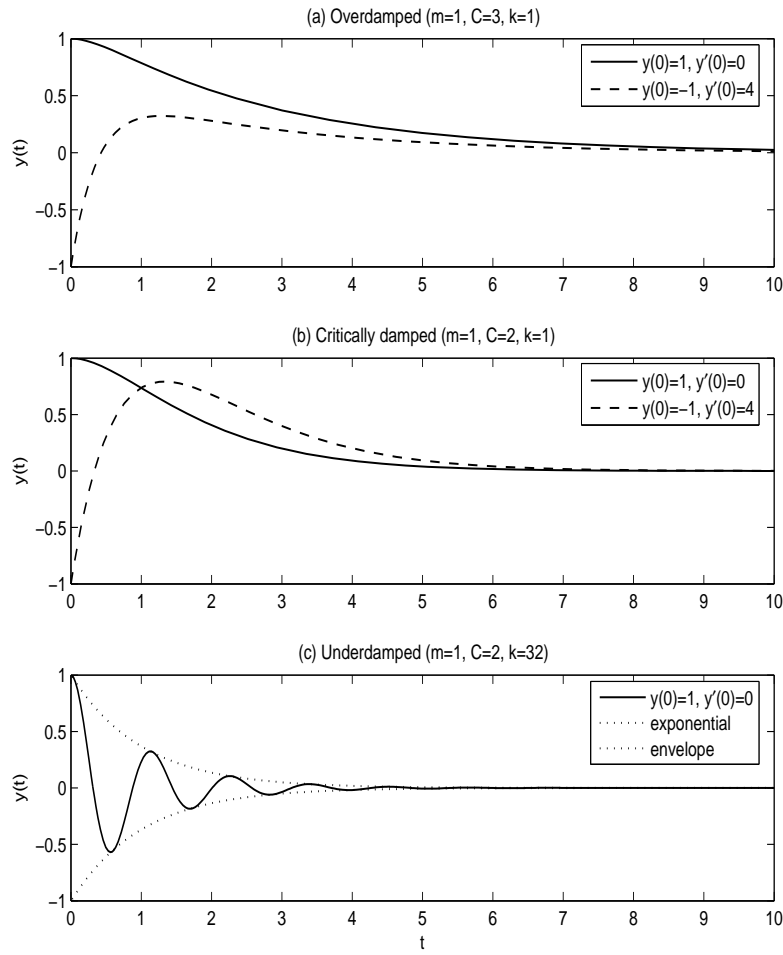


FIGURE 2.1. Overdamping, critical damping and underdamping for a simple mass–spring system. We used the specific values for  $m$ ,  $C$  and  $k$  shown. In (a) and (b) we plotted the two solutions corresponding to the two distinct sets of initial conditions shown.



## Non-homogeneous linear ODEs

### 3.1. Example applications

**3.1.1. Forced spring systems.** What happens if our spring system (damped or undamped) is forced externally? For example, consider the following initial value problem for a forced harmonic oscillator (which models a mass on the end of a spring which is forced externally)

$$m \frac{d^2 y}{dt^2} + C \frac{dy}{dt} + ky = f(t), \quad y(0) = y'(0) = 0.$$

Here  $y(t)$  is the displacement of the mass,  $m$ , from equilibrium at time  $t$ . The external forcing  $f(t)$  could be oscillatory, say

$$f(t) = A \sin \omega t,$$

where  $A$  and  $\omega$  are also (given) positive constants. We will see in this chapter how solutions to such problems can behave quite dramatically when the frequency of the external force  $\omega$  matches that of the natural oscillations  $\omega_0 = +\sqrt{k/m}$  of the undamped ( $C \equiv 0$ ) system—undamped resonance! We will also discuss the phenomenon of resonance in the presence of damping ( $C > 0$ ).

**3.1.2. Electrical circuits.** Consider a simple series circuit (see Figure A.1 in Appendix A.2.1) which has a resistor with resistance  $R$ , a capacitor of capacitance  $C$ , an inductor of inductance  $\ell$  (all positive constants) and a battery which provides an impressed voltage  $V(t)$ . The total charge  $Q(t)$  in such a circuit is modelled by the ODE

$$\ell \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q = V(t). \quad (3.1)$$

‘Feedback-squeals’ in electric circuits at concerts are an example of resonance effects in such equations.

### 3.2. Linear operators

**3.2.1. Concept.** Consider the general non-homogeneous second order linear ODE

$$a(t) \frac{d^2 y}{dt^2} + b(t) \frac{dy}{dt} + c(t)y = f(t). \quad (3.2)$$

We can abbreviate the ODE (3.2) to

$$Ly(t) = f(t), \quad (3.3)$$

where  $L$  is the *differential operator*

$$L = a(t) \frac{d^2}{dt^2} + b(t) \frac{d}{dt} + c(t). \quad (3.4)$$

We can re-interpret our general linear second order ODE as follows. When we operate on a function  $y(t)$  by the differential operator  $L$ , we generate a new function of  $t$ :

$$Ly(t) = a(t)y''(t) + b(t)y'(t) + c(t)y(t).$$

To solve (3.3), we want the most general expression,  $y$  as a function of  $t$ , which is such that  $L$  operated on  $y$  gives  $f(t)$ .

**3.2.2. Definition.** A linear operator is defined by the following property.

An operator  $L$  is said to be *linear* if

$$L(\alpha y_1 + \beta y_2) = \alpha Ly_1 + \beta Ly_2,$$

for every  $y_1$  and  $y_2$ , and all constants  $\alpha$  and  $\beta$ .

**3.2.3. Example.** Show that the operator  $L$  in (3.4) is linear.

### 3.3. Solving non-homogeneous linear ODEs

Consider the linear second order ODE

$$Ly = f. \tag{3.5}$$

To solve this problem we first consider the solution to the associated homogeneous ODE (called the *Complementary Function*):

$$Ly_{CF} = 0. \tag{3.6}$$

Since this ODE (3.6) is linear, second order and homogeneous, we can *always* find an expression for the solution—in the constant coefficient case the solution has one of the forms given in Table 2.2.7. Now suppose that we can find a particular solution—often called the *particular integral (PI)*—of (3.5), i.e. some function,  $y_{PI}$ , that satisfies (3.5):

$$Ly_{PI} = f.$$

Then the complete, *general solution* of (3.5) is

$$y = y_{CF} + y_{PI}.$$

This must be the general solution because it contains two arbitrary constants (in the  $y_{CF}$  part) and satisfies the ODE, since, using that  $L$  is a linear operator,

$$L(y_{CF} + y_{PI}) = \underbrace{Ly_{CF}}_{=0} + \underbrace{Ly_{PI}}_{=f} = f.$$

Hence to summarize:

To solve a non-homogenous linear ODE of the form

$$Ly = f ,$$

- (1) Find the general solution—the complementary function  $y_{CF}$ —to the corresponding homogeneous linear ODE

$$Ly_{CF} = 0 .$$

- (2) Find any solution—the particular integral  $y_{PI}$ —to the full non-homogeneous linear ODE

$$Ly_{PI} = f .$$

Then the general solution to the non-homogeneous linear ODE is

$$y = y_{CF} + y_{PI} .$$

### 3.4. Method of undetermined coefficients

We now need to know how to obtain a particular integral. For special cases of the inhomogeneity  $f(t)$  we use the *method of undetermined coefficients*, though there is a more general method called the *method of variation of parameters*—see for example Kreyszig [8]. In the *method of undetermined coefficients* we make an initial assumption about the form of the particular integral  $y_{PI}$ , but with the coefficients left unspecified. We substitute our guess for  $y_{PI}$  into the linear ODE,  $Ly = f$ , and attempt to determine the coefficients so that  $y_{PI}$  satisfies the equation.

**3.4.1. Example.** Find the general solution of the linear ODE

$$y'' - 3y' - 4y = 3e^{2t} .$$

**3.4.2. Solution.**

*Step 1: Find the complementary function.* Looking for a solution of the form  $e^{\lambda t}$ , the auxiliary equation is  $\lambda^2 - 3\lambda - 4 = 0$  which has two real distinct roots  $\lambda_1 = 4$  and  $\lambda_2 = -1$ , hence from Table 2.2.7, we have

$$y_{CF}(t) = C_1 e^{4t} + C_2 e^{-t} .$$

*Step 2: Find the particular integral.* Assume that the particular integral has the form (using Table 3.4.2)

$$y_{PI}(t) = Ae^{2t} ,$$

where the coefficient  $A$  is yet to be determined. Substituting this form for  $y_{PI}$  into the ODE, we get

$$(4A - 6A - 4A)e^{2t} = 3e^{2t} \quad \Leftrightarrow \quad -6Ae^{2t} = 3e^{2t} .$$

Hence  $A$  must be  $-\frac{1}{2}$  and a particular solution is

$$y_{PI}(t) = -\frac{1}{2}e^{2t} .$$

| Inhomogeneity $f(t)$                       | Try $y_{PI}(t)$   |
|--|---|
| $e^{\alpha t}$                             | $Ae^{\alpha t}$   |
| $\sin(\alpha t)$                           | $A \sin(\alpha t) + B \cos(\alpha t)$                       |
| $\cos(\alpha t)$                           | $A \sin(\alpha t) + B \cos(\alpha t)$                       |
| $b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n$ | $A_0 + A_1 t + A_2 t^2 + \cdots + A_n t^n$                  |
| $e^{\alpha t} \sin(\beta t)$               | $Ae^{\alpha t} \sin(\beta t) + Be^{\alpha t} \cos(\beta t)$ |
| $e^{\alpha t} \cos(\beta t)$               | $Ae^{\alpha t} \sin(\beta t) + Be^{\alpha t} \cos(\beta t)$ |

TABLE 3.1. *Method of undetermined coefficients.* When the inhomogeneity  $f(t)$  has the form (or is any constant multiplied by this form) shown in the left-hand column, then you should try a  $y_{PI}(t)$  of the form shown in the right-hand column. We can also make the obvious extensions for combinations of the inhomogeneities  $f(t)$  shown.

Hence the general solution to the differential equation is

$$y(t) = \underbrace{C_1 e^{4t} + C_2 e^{-t}}_{y_{CF}} - \underbrace{\frac{1}{2} e^{2t}}_{y_{PI}}.$$

**3.4.3. Example.** Find the general solution of the linear ODE

$$y'' - 3y' - 4y = 2 \sin t.$$

**3.4.4. Solution.**

*Step 1: Find the complementary function.* In this case, the complementary function is clearly the same as in the last example—the corresponding homogeneous equation is the same—hence

$$y_{CF}(t) = C_1 e^{4t} + C_2 e^{-t}.$$

*Step 2: Find the particular integral.* Assume that  $y_{PI}$  has the form (using Table 3.4.2)

$$y_{PI}(t) = A \sin t + B \cos t,$$

where the coefficients  $A$  and  $B$  are yet to be determined. Substituting this form for  $y_{PI}$  into the ODE implies

$$\begin{aligned} (-A \sin t - B \cos t) - 3(A \cos t - B \sin t) - 4(A \sin t + B \cos t) &= 2 \sin t \\ \Leftrightarrow (-A + 3B - 4A) \sin t + (-B - 3A - 4B) \cos t &= 2 \sin t. \end{aligned}$$

Equating coefficients of  $\sin t$  and also  $\cos t$ , we see that

$$-5A + 3B = 2 \quad \text{and} \quad -5B - 3A = 0.$$

Hence  $A = -\frac{5}{17}$  and  $B = \frac{3}{17}$  and so

$$y_{\text{PI}}(t) = -\frac{5}{17} \sin t + \frac{3}{17} \cos t.$$

Thus the general solution is

$$y(t) = \underbrace{C_1 e^{4t} + C_2 e^{-t}}_{y_{\text{CF}}} + \underbrace{-\frac{5}{17} \sin t + \frac{3}{17} \cos t}_{y_{\text{PI}}}.$$

### 3.5. Degenerate inhomogeneities

**3.5.1. Example.** Find the general solution of the degenerate linear ODE

$$y'' + 4y = 3 \cos 2t.$$

#### 3.5.2. Solution.

*Step 1: Find the complementary function.* First we solve the corresponding homogeneous equation

$$y''_{\text{CF}} + 4y_{\text{CF}} = 0, \quad (3.7)$$

to find the complementary function. Two solutions to this equation are  $\sin 2t$  and  $\cos 2t$ , and so the complementary function is

$$y_{\text{CF}}(t) = C_1 \sin 2t + C_2 \cos 2t,$$

where  $C_1$  and  $C_2$  are arbitrary constants.

*Step 2: Find the particular integral.* Using Table 3.4.2, assume that  $y_{\text{PI}}$  has the form

$$y_{\text{PI}}(t) = A \sin 2t + B \cos 2t,$$

where the coefficients  $A$  &  $B$  are yet to be determined. Substituting this form for  $y_{\text{PI}}$  into the ODE implies

$$\begin{aligned} (-4A \sin 2t - 4B \cos 2t) + 4(A \sin 2t + B \cos 2t) &= 3 \cos 2t \\ \Leftrightarrow (4B - 4B) \sin 2t + (4A - 4A) \cos 2t &= 3 \cos 2t. \end{aligned}$$

Since the left-hand side is zero, there is no choice of  $A$  and  $B$  that satisfies this equation. Hence for some reason we made a poor initial choice for our particular solution  $y_{\text{PI}}(t)$ . This becomes apparent when we recall the solutions to the homogeneous equation (3.7) are  $\sin 2t$  and  $\cos 2t$ . These are solutions to the homogeneous equation and cannot possibly be solutions to the non-homogeneous case we're considering. We must therefore try a slightly different choice for  $y_{\text{PI}}(t)$ , for example,

$$y_{\text{PI}}(t) = At \cos 2t + Bt \sin 2t.$$

Substituting this form for  $y_{\text{PI}}$  into the ODE and cancelling terms implies

$$-4A \sin 2t + 4B \cos 2t = 3 \cos 2t$$

Therefore, equating coefficients of  $\sin 2t$  and  $\cos 2t$ , we see that  $A = 0$  and  $B = \frac{3}{4}$  and so

$$y_{\text{PI}}(t) = \frac{3}{4}t \sin 2t.$$

Hence the general solution is

$$y(t) = \underbrace{C_1 \sin 2t + C_2 \cos 2t}_{y_{\text{CF}}} + \underbrace{\frac{3}{4}t \sin 2t}_{y_{\text{PI}}}.$$

Occasionally such a modification, will be insufficient to remove all duplications of the solutions of the homogeneous equation, in which case it is necessary to multiply by  $t$  a second time. For a second order equation though, it will never be necessary to carry the process further than two modifications.

**3.5.3. Example.**  $y'' - 2y' - 3y = 3e^{3t}$ .

**3.5.4. Example.**  $y'' - 6y' + 9y = 3e^{3t}$ .

### 3.6. Initial and boundary value problems

- *Initial value problems (IVPs):* given values for the solution,  $y(t_0) = y_0$ , and its derivative,  $y'(t_0) = v_0$ , at a given time  $t = t_0$ , are used to determine the solution.
- *Boundary value problems (BVPs):* given values for the solution,  $y(t_0) = y_0$  and  $y(t_1) = y_1$ , at two distinct times  $t = t_0$  and  $t = t_1$ , are used to determine the solution.

In either case the two pieces of information given (either the initial or boundary data) are used to determine the specific values for the arbitrary constants in the general solution which generate the specific solution satisfying that (initial or boundary) data.

**3.6.1. Example (initial value problem).** Find the solution to the initial value problem

$$y'' - 3y' - 4y = 2 \sin t, \quad y(0) = 1, \quad y'(0) = 2.$$

### 3.7. Resonance

**3.7.1. Undamped resonance.** Consider the following initial value problem for a forced harmonic oscillator, which for example, models a mass on the end of a spring which is forced externally,

$$y'' + \omega_0^2 y = \frac{1}{m} f(t), \quad y(0) = y'(0) = 0.$$

Here  $y(t)$  is the displacement of the mass  $m$  from equilibrium at time  $t$ , and  $\omega_0 = +\sqrt{k/m}$  is a positive constant representing the natural frequency of oscillation when no forcing is present. Suppose

$$f(t) = A \sin \omega t$$

is the external oscillatory forcing, where  $A$  and  $\omega$  are also positive constants. Assume for the moment that  $\omega \neq \omega_0$ .

We proceed by first finding a solution to the corresponding homogeneous problem,

$$\begin{aligned} y''_{\text{CF}} + \omega_0^2 y_{\text{CF}} &= 0 \\ \Rightarrow y_{\text{CF}}(t) &= C_1 \cos \omega_0 t + C_2 \sin \omega_0 t, \end{aligned} \tag{3.8}$$

where  $C_1$  and  $C_2$  are arbitrary constants. Next we look for a particular integral, using Table 3.1 we should try

$$y_{\text{PI}}(t) = D_1 \cos \omega t + D_2 \sin \omega t, \tag{3.9}$$

where  $D_1$  and  $D_2$  are the constants to be determined. Substituting this trial particular integral into the full non-homogeneous solution we get,

$$-\omega^2(D_1 \cos \omega t + D_2 \sin \omega t) + \omega_0^2(D_1 \cos \omega t + D_2 \sin \omega t) = \frac{1}{m} A \sin \omega t.$$

Equating coefficients of  $\cos \omega t$  and  $\sin \omega t$ , we see that

$$-\omega^2 D_1 + \omega_0^2 D_1 = 0 \quad \text{and} \quad -\omega^2 D_2 + \omega_0^2 D_2 = \frac{1}{m} A.$$

Hence, provided  $\omega \neq \omega_0$ , then  $D_1 = 0$  and  $D_2 = A/m(\omega_0^2 - \omega^2)$ , so that

$$y_{\text{PI}}(t) = \frac{A}{m(\omega_0^2 - \omega^2)} \sin \omega t.$$

Hence the general solution is

$$y(t) = \underbrace{C_1 \cos \omega_0 t + C_2 \sin \omega_0 t}_{y_{\text{CF}}} + \underbrace{\frac{A}{m(\omega_0^2 - \omega^2)} \sin \omega t}_{y_{\text{PI}}}.$$

Now using the initial conditions,

$$y(0) = 0 \quad \Rightarrow \quad C_1 = 0,$$

while,

$$y'(0) = 0 \quad \Rightarrow \quad \omega_0 C_2 + \omega \cdot \frac{A}{m(\omega_0^2 - \omega^2)} = 0 \quad \Rightarrow \quad C_2 = -\frac{A\omega}{\omega_0 m(\omega_0^2 - \omega^2)}.$$

Thus finally the solution to the initial value problem is given by

$$y(t) = \frac{A\omega}{m(\omega^2 - \omega_0^2)} \cdot \left( \underbrace{\frac{1}{\omega_0} \sin \omega_0 t}_{\text{natural oscillation}} - \underbrace{\frac{1}{\omega} \sin \omega t}_{\text{forced oscillation}} \right), \quad (3.10)$$

where the first oscillatory term represents the natural oscillations, and the second, the forced mode of vibration.

What happens when  $\omega \rightarrow \omega_0$ ? If we naively take the limit  $\omega \rightarrow \omega_0$  in (3.10) we see that the two oscillatory terms combine to give zero, but also, the denominator in the multiplicative term  $\frac{A\omega}{m(\omega^2 - \omega_0^2)}$  also goes to zero. This implies we should be much more careful about taking this limit. Let's rewrite this problematic ratio as follows:

$$\begin{aligned} \frac{\frac{1}{\omega_0} \sin \omega_0 t - \frac{1}{\omega} \sin \omega t}{\omega^2 - \omega_0^2} &= \frac{\frac{1}{\omega_0} \sin \omega_0 t - \frac{1}{\omega} \sin \omega t}{(\omega + \omega_0)(\omega - \omega_0)} \\ &= \frac{1}{\omega(\omega + \omega_0)} \cdot \frac{\omega \frac{1}{\omega_0} \sin \omega_0 t - \sin \omega t}{\omega - \omega_0} \\ &= \frac{1}{\omega(\omega + \omega_0)} \cdot \frac{\omega \frac{1}{\omega_0} \sin \omega_0 t - \sin \omega_0 t + \sin \omega_0 t - \sin \omega t}{\omega - \omega_0} \\ &= \frac{1}{\omega(\omega + \omega_0)} \cdot \frac{(\omega - \omega_0) \frac{1}{\omega_0} \sin \omega_0 t - (\sin \omega t - \sin \omega_0 t)}{\omega - \omega_0} \\ &= \frac{1}{\omega(\omega + \omega_0)} \cdot \left( \frac{1}{\omega_0} \sin \omega_0 t - \underbrace{\frac{\sin \omega t - \sin \omega_0 t}{\omega - \omega_0}}_{\text{careful limit}} \right). \end{aligned} \quad (3.11)$$

Now the limit issue is isolated to the second term on the right-hand side. The idea now is to imagine the function  $F(\omega) = \sin \omega t$  as a function of  $\omega$ , with time  $t$  as a constant parameter and to determine the limit

$$\lim_{\omega \rightarrow \omega_0} \frac{F(\omega) - F(\omega_0)}{\omega - \omega_0},$$

which in fact, by definition, is nothing other than  $F'(\omega_0)$ !

Hence when  $F(\omega) = \sin \omega t$ , we have that

$$\lim_{\omega \rightarrow \omega_0} \frac{\sin \omega t - \sin \omega_0 t}{\omega - \omega_0} = t \cos \omega_0 t.$$

Thus if we take the limit  $\omega \rightarrow \omega_0$  in (3.11) we see that

$$\lim_{\omega \rightarrow \omega_0} \frac{1}{\omega(\omega + \omega_0)} \cdot \left( \frac{1}{\omega_0} \sin \omega_0 t - \frac{\sin \omega t - \sin \omega_0 t}{\omega - \omega_0} \right) = \frac{1}{2\omega_0^2} \cdot \left( \frac{1}{\omega_0} \sin \omega_0 t - t \cos \omega_0 t \right).$$

Hence the solution to the initial value problem when  $\omega = \omega_0$  is

$$y(t) = \frac{A}{2m\omega_0} \cdot \left( \underbrace{\frac{1}{\omega_0} \sin \omega_0 t}_{\text{natural oscillation}} - \underbrace{t \cos \omega_0 t}_{\text{resonant term}} \right). \quad (3.12)$$

The important aspect to notice is that when  $\omega = \omega_0$ , the second term ‘ $t \cos \omega_0 t$ ’ grows without bound (the amplitude of these oscillations grows like  $t$ ) and this is the *signature of undamped resonance*.

**3.7.2. Damped resonance.** Now suppose we introduce damping into our simple spring system so that the coefficient of friction  $C > 0$ . In the overdamped, critically damped or underdamped cases the complementary function is always exponentially decaying in time. We call this part of the solution the *transient* solution—it will be significant initially, but it decays to zero exponentially fast. The contribution to the solution from the particular integral which arises from the external forcing, cannot generate unbounded resonant behaviour for any bounded driving oscillatory force. However the amplitude of the forced oscillations of the solution (from the particular integral, which is potentially the only significant component for large times) does have a global maximum at a given *practical resonance* frequency. This is covered in detail in many engineering books, for example Kreyszig [8] p. 113.

### 3.8. Equidimensional equations

A linear differential equation of the form

$$ax^2 \frac{d^2 y}{dx^2} + bx \frac{dy}{dx} + cy = f(x), \quad (3.13)$$

with  $a$ ,  $b$  &  $c$  all constant, is an example of an *equidimensional* or *Euler-Cauchy* equation of second order.

To solve equations of this form, introduce a new independent variable

$$z = \log x \quad \Leftrightarrow \quad x = e^z.$$

Then the chain rule implies

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x}, \quad \text{i.e.} \quad x \frac{dy}{dx} = \frac{dy}{dz}. \quad (3.14)$$

Further,

$$\begin{aligned} \frac{d^2 y}{dx^2} &= \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{dy}{dz} \cdot \frac{1}{x} \right) = \frac{d}{dx} \left( \frac{dy}{dz} \right) \cdot \frac{1}{x} - \frac{dy}{dz} \cdot \frac{1}{x^2} \\ &= \frac{d}{dz} \left( \frac{dy}{dz} \right) \cdot \frac{dz}{dx} \cdot \frac{1}{x} - \frac{dy}{dz} \cdot \frac{1}{x^2} \\ &= \left( \frac{d^2 y}{dz^2} - \frac{dy}{dz} \right) \cdot \frac{1}{x^2}, \end{aligned}$$

i.e.

$$x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz}. \quad (3.15)$$

If we substitute (3.14) and (3.15) into the equidimensional equation (3.13), we get

$$a \frac{d^2 y}{dz^2} + (b - a) \frac{dy}{dz} + cy = f(e^z). \quad (3.16)$$

Now to solve (3.16), we can use the techniques we have learned to solve constant coefficient linear second order ODEs. Once you have solved (3.16), remember to substitute back that  $z = \log x$ .

**3.8.1. Example.** Find the general solution to

$$x^2 y'' - 2xy' + 2y = (\ln(x))^2 - \ln(x^2).$$

**3.8.2. Remark.** That such equations are called *equidimensional* refers to the fact that they are characterized by the property that the linear operator on the left-hand side

$$L \equiv ax^2 \frac{d^2}{dx^2} + bx \frac{d}{dx} + c$$

is invariant under the transformation  $x \rightarrow kx$ .

**3.8.3. Example (Black–Scholes).** In 1973 Black & Scholes derived the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0,$$

where  $r$  and  $\sigma$  are constants representing the risk-free interest and volatility of the underlying traded index, respectively. Here  $S$  is the price (which varies stochastically) of the underlying traded index and  $V$  is the value of a financial option on that index. Myron Scholes won the *Nobel Prize* in 1997 for his work involving this partial differential equation!

Notice that it is an equidimensional equation with respect to  $S$ , and is solved using the same change of variables we performed above. (This equation is also known—in different guises—as the Fokker-Planck or backward Kolmogorov equation.)

### Summary: solving linear constant coefficient second order IVPs

In general, to solve the linear second order non-homogeneous constant coefficient ordinary differential initial value problem,

$$a \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + cy = f(t), \quad \text{with } y(0) = \alpha, \quad y'(0) = \beta, \quad (3.17)$$

where  $a$ ,  $b$  and  $c$  are given intrinsic constants, and  $\alpha$  and  $\beta$  are given initial data, proceed as follows.

- *Step 1: Find the complementary function* i.e. find the general solution to the associated homogeneous ODE

$$Ly_{CF} = 0. \quad (3.18)$$

To achieve this, try to find a solution to (3.18) of the form  $y_{CF} = e^{\lambda t}$ . This generates the *auxiliary equation*

$$a\lambda^2 + b\lambda + c = 0.$$

Then pick the solution given in Table 2.2.7 depending on whether  $b^2 - 4ac$  is positive, zero or negative. This solution always has the form

$$y_{CF}(t) = C_1 y_1(t) + C_2 y_2(t),$$

where  $C_1$  and  $C_2$  are arbitrary constants.

- *Step 2: Find the particular integral* i.e. find any solution  $y_{PI}$  of the full non-homogeneous equation (3.17),

$$Ly_{PI} = f,$$

by using the method of undetermined coefficients—refer to Table 3.4.2.

Then the *general solution* of (3.17) is

$$y(t) = y_{CF} + y_{PI} \quad (3.19)$$

$$\Rightarrow y(t) = C_1 y_1(t) + C_2 y_2(t) + y_{PI}(t). \quad (3.20)$$

- *Step 3: Use the initial conditions to determine the arbitrary constants* Using the first initial condition, and then differentiating the general solution (3.20) and substituting in the second initial condition we get, respectively,

$$\alpha = C_1 y_1(0) + C_2 y_2(0) + y_{PI}(0), \quad (3.21a)$$

$$\beta = C_1 y_1'(0) + C_2 y_2'(0) + y_{PI}'(0). \quad (3.21b)$$

Now solve the simultaneous equations (3.21) for  $C_1$  and  $C_2$  and substitute these values into (3.20).

## Laplace transforms

### 4.1. Introduction

**4.1.1. Example.** Consider a damped spring system which consists of a mass which slides on a horizontal surface, and is attached to a spring, which is fixed to a vertical wall at the other end (see Figure 1.1 in Chapter 1). Suppose that the mass, initially at rest in the equilibrium position, is given a sharp hammer blow at time  $t_0 > 0$ , so that the equation of motion and initial conditions for the mass are

$$y'' + 3y' + 2y = \delta(t - t_0), \quad \text{with } y(0) = y'(0) = 0. \quad (4.1)$$

Here the external forcing function is  $f(t) = \delta(t - t_0)$ . The Dirac delta function,  $\delta(t - t_0)$ , is supposed to represent the action of a force acting instantaneously at the time  $t_0$  and imparting a unit impulse (momentum) to the mass. The method of Laplace transforms is ideally suited to dealing with such situations and can be used to determine the solution to such initial value problems very conveniently.

**4.1.2. Definition (of Laplace transform).** Formally:

Suppose the function  $f(t)$  is defined for all  $t \geq 0$ . The *Laplace transform* of  $f(t)$  is defined, as a function of the variable  $s$  by the integral,

$$\bar{f}(s) = \mathcal{L}\{f(t)\} \equiv \int_0^{\infty} e^{-st} f(t) dt.$$

$\bar{f}(s)$  is defined for those values of  $s$  for which the right-hand integral is finite.

**4.1.3. Example.** For any  $s > 0$ ,  $\mathcal{L}\{1\} = \int_0^{\infty} e^{-st} dt = \left[ -\frac{e^{-st}}{s} \right]_0^{\infty} = \frac{1}{s}$ .

**4.1.4. Example.** For any  $s > a$ ,  $\mathcal{L}\{e^{at}\} = \int_0^{\infty} e^{-st} e^{at} dt = \int_0^{\infty} e^{-(s-a)t} dt = \frac{1}{s-a}$ .

**4.1.5. Example.** For any  $s > 0$ ,  $\mathcal{L}\{\sin at\} = \int_0^{\infty} e^{-st} \sin at dt = \frac{a}{s^2 + a^2}$ .

**4.1.6. Example.** For any  $s > 0$ ,  $\mathcal{L}\{\cos at\} = \int_0^{\infty} e^{-st} \cos at dt = \frac{s}{s^2 + a^2}$ .

## 4.2. Properties of Laplace transforms

**4.2.1. Basic properties.** The essential properties can be summarized as follows.

Suppose  $f(t)$  and  $g(t)$  are any two functions with Laplace transforms  $\bar{f}(s)$  and  $\bar{g}(s)$ , respectively, and that  $a$  and  $b$  are any two constants.

- $\mathcal{L}$  is a *linear integral operator*.

$$\mathcal{L}\{af(t) + bg(t)\} = a\bar{f}(s) + b\bar{g}(s).$$

- *Derivative theorem*.

$$\mathcal{L}\{f'(t)\} = s\bar{f}(s) - f(0),$$

$$\mathcal{L}\{f''(t)\} = s^2\bar{f}(s) - sf(0) - f'(0),$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0).$$

- *Shift theorem*.

$$\mathcal{L}\{e^{-at}f(t)\} = \bar{f}(s + a).$$

- *Second shift theorem*. If

$$f(t) = \begin{cases} g(t - a), & t \geq a, \\ 0, & t < a, \end{cases}$$

then

$$\mathcal{L}\{f(t)\} = e^{-sa}\bar{g}(s).$$

- *Convolution theorem*. If we define the convolution product of two functions to be

$$f(t) * g(t) \equiv \int_{-\infty}^{+\infty} f(\tau)g(t - \tau) d\tau \equiv \int_{-\infty}^{+\infty} f(t - \tau)g(\tau) d\tau,$$

then

$$\mathcal{L}\{f(t) * g(t)\} = \bar{f}(s) \cdot \bar{g}(s).$$

**4.2.2. Example (Derivative theorem).** Formally, using the definition of the Laplace transform and then integrating by parts

$$\begin{aligned} \mathcal{L}\{f'(t)\} &= \int_0^{\infty} e^{-st} f'(t) dt \\ &= [e^{-st} f(t)]_0^{\infty} + s \int_0^{\infty} e^{-st} f(t) dt \\ &= -f(0) + s\bar{f}(s). \end{aligned}$$

**4.2.3. Example (Derivative theorem application).** We know that the Laplace transform of  $f(t) = \sin at$  is

$$\bar{f}(s) = \frac{a}{s^2 + a^2}.$$

Hence to find the Laplace transform of  $g(t) = \cos at$ , we note that

$$g(t) = \frac{1}{a}f'(t) \quad \Rightarrow \quad \bar{g}(s) = \frac{1}{a} \cdot \frac{sa}{s^2 + a^2} - \frac{1}{a} \sin 0 = \frac{s}{s^2 + a^2}.$$

**4.2.4. Example (Shift theorem).** Using the Shift Theorem,

$$\mathcal{L}\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2}.$$

i.e. the function is *shifted* in the transform space when it is multiplied by  $e^{at}$  in the non-transform space.

**4.2.5. Example (Second shift theorem).** Consider

$$f(t) = \begin{cases} g(t-a), & t \geq a, \\ 0, & t < a, \end{cases}$$

with  $a > 0$  constant. i.e.  $f(t)$  is the function  $g$  shifted to the right along the real axis by a distance  $a$ . Setting  $u = t - a$  we see that

$$\begin{aligned} \bar{f}(s) &= \int_0^\infty e^{-st} g(t-a) dt \\ &= \int_0^\infty e^{-sa} e^{-su} g(u) du \\ &= e^{-sa} \int_0^\infty e^{-su} g(u) du \\ &= e^{-sa} \bar{g}(s). \end{aligned}$$

### 4.3. Solving linear constant coefficients ODEs via Laplace transforms

**4.3.1. Example.** Find the solution to the following initial value problem:

$$y'' + 4y' + 8y = 1, \quad y(0) = y'(0) = 0.$$

**4.3.2. Solution.** Take the Laplace transform of both sides of the ODE, we have that

$$\begin{aligned} &\mathcal{L}\{y''(t) + 4y'(t) + 8y(t)\} = \mathcal{L}\{1\} \\ \Leftrightarrow &\mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y'(t)\} + 8\mathcal{L}\{y(t)\} = \mathcal{L}\{1\} \\ \Leftrightarrow &s^2\bar{y}(s) - sy(0) - y'(0) + 4(s\bar{y}(s) - y(0)) + 8\bar{y}(s) = \frac{1}{s} \\ \Leftrightarrow &(s^2 + 4s + 8)\bar{y}(s) - (s + 4)y(0) - y'(0) = \frac{1}{s} \\ \Leftrightarrow &(s^2 + 4s + 8)\bar{y}(s) = \frac{1}{s}, \end{aligned}$$

where in the last step we have used that  $y(0) = y'(0) = 0$  for this problem. Now look at this last equation—notice that by taking the Laplace transform of the differential equation for  $y(t)$ , we have converted it to an algebraic equation for  $\bar{y}(s)$ . This linear algebraic equation can be easily solved:

$$\bar{y}(s) = \frac{1}{s(s^2 + 4s + 8)}.$$

Hence we now know what the Laplace transform of the solution of the differential equation (plus initial conditions) looks like. The question now is, knowing  $\bar{y}(s)$ , can we figure out what  $y(t)$  is?

We use partial fractions to split up  $\bar{y}(s)$  as follows, i.e. we seek to write  $\bar{y}(s)$  in the form (the idea is to try to split up  $\bar{y}(s)$  into simpler parts we can handle)

$$\frac{1}{s(s^2 + 4s + 8)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 8}. \quad (4.2)$$

The question is, can we find constants  $A$ ,  $B$  and  $C$  so that this last expression is true for all  $s \neq 0$ ? Multiply both sides of the equation by the denominator on the left-hand side; this gives

$$\begin{aligned} 1 &= A(s^2 + 4s + 8) + (Bs + C)s \\ \Leftrightarrow 1 &= (A + B)s^2 + (4A + C)s + 8A. \end{aligned}$$

We want this to hold for all  $s \neq 0$ . Hence equating powers of  $s$  we see that

$$\begin{aligned} s^0: &\Rightarrow 1 = 8A \quad \Rightarrow A = 1/8, \\ s^1: &\Rightarrow 0 = 4A + C \quad \Rightarrow C = -1/2, \\ s^2: &\Rightarrow 0 = A + B \quad \Rightarrow B = -1/8. \end{aligned}$$

Hence

$$\bar{y}(s) = \frac{1}{8} \cdot \frac{1}{s} - \frac{1}{8} \cdot \frac{s + 4}{s^2 + 4s + 8}.$$

Completing the square for the denominator in the second term we see that

$$\begin{aligned} \bar{y}(s) &= \frac{1}{8} \cdot \frac{1}{s} - \frac{1}{8} \cdot \frac{s + 4}{((s + 2)^2 + 4)} \\ \Leftrightarrow \bar{y}(s) &= \frac{1}{8} \cdot \frac{1}{s} - \frac{1}{8} \cdot \frac{s + 2}{(s + 2)^2 + 4} - \frac{1}{8} \cdot \frac{2}{(s + 2)^2 + 4} \\ \Leftrightarrow y(t) &= \frac{1}{8} - \frac{1}{8}e^{-2t} \cos 2t - \frac{1}{8}e^{-2t} \sin 2t, \end{aligned}$$

using the table of Laplace transforms in the last step.

**4.3.3. Example.** Solve the following initial value problem using the method of Laplace transforms

$$y'' + 4y' + 4y = 6e^{-2t}, \quad y(0) = -2, \quad y'(0) = 8.$$

#### 4.4. Impulses and Dirac's delta function

**4.4.1. Impulse.** Laplace transforms are particularly useful when we wish to solve a differential equation which models a mechanical or electrical system and which involves an impulsive force or current. For example, if a mechanical system is given a blow by a hammer. In mechanics, the *impulse*  $I(t)$  of a force  $f(t)$  which acts over a given time interval, say  $t_0 \leq t \leq t_1$ , is defined to be

$$I(t) = \int_{t_0}^{t_1} f(t) dt.$$

It represents the total momentum imparted to the system over the time interval  $t_0 \leq t \leq t_1$  by  $f(t)$ . For an electrical circuit the analogous quantity is obtained by replacing  $f(t)$  by the electromotive force (applied voltage)  $V(t)$ .

**4.4.2. Dirac's delta function.** Let's suppose we apply a force

$$f_\epsilon(t) = \begin{cases} 1/\epsilon, & \text{if } t_0 \leq t \leq t_0 + \epsilon, \\ 0, & \text{otherwise,} \end{cases}$$

i.e. we apply a constant force  $1/\epsilon$  over the time interval  $t_0 \leq t \leq t_0 + \epsilon$ , where  $\epsilon \ll 1$  is a small parameter and  $t_0 > 0$ . Then the total impulse, for all  $t \geq 0$ , corresponding to this force is

$$I_\epsilon(t) = \int_0^\infty f_\epsilon(t) dt = \int_{t_0}^{t_0+\epsilon} \frac{1}{\epsilon} dt = 1.$$

Note that  $I_\epsilon(t)$  represents the area under the graph of  $f_\epsilon(t)$ , and this last result shows that this area is independent of  $\epsilon$ . Hence if we take the limit as  $\epsilon \rightarrow 0$ , then

$$f_\epsilon(t) \rightarrow \delta(t - t_0),$$

where  $\delta(t - t_0)$  is called the *Dirac delta function*. It has the property that

$$\int_0^\infty \delta(t - t_0) dt = 1,$$

and that it is zero everywhere except at  $t = t_0$ , where it is undefined. In fact it is not really a function at all, but is an example of a *generalized function* or (*singular*) *distribution*. Of interest to us here is that it represents an impulse, of magnitude 1, acting over an infinitesimally short time interval, exactly as a hammer hitting a mass and imparting some momentum to it (via an impulse). In particular,  $\delta(t - t_0)$  has a simple Laplace transform. First, let's consider the Laplace transform of  $f_\epsilon(t)$ :

$$\mathcal{L}\{f_\epsilon(t)\} = \int_0^\infty f_\epsilon(t) e^{-st} dt = \int_{t_0}^{t_0+\epsilon} \frac{1}{\epsilon} e^{-st} dt = e^{-st_0} \cdot \frac{1 - e^{-\epsilon s}}{\epsilon s}.$$

Now taking the limit  $\epsilon \rightarrow 0$  in this last expression, we get

$$\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}.$$

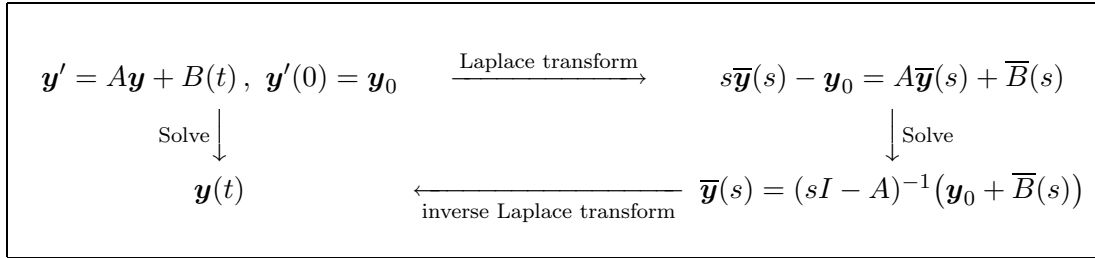
Hence the Dirac delta function is much more easily handled in the Laplace transform space, where it is represented by an ordinary exponential function, as opposed to its generalized function guise in the non-transform space. Note also that if we take the limit  $t_0 \rightarrow 0$  we get that  $\mathcal{L}\{\delta(t)\} = 1$ .

**4.4.3. Example.** Consider the damped spring system shown in Figure 1.1. Suppose that the mass, initially at rest in the equilibrium position, is given a sharp hammer blow at time  $t_0 > 0$ , so that the equation of motion and initial conditions for the mass are

$$y'' + 3y' + 2y = \delta(t - t_0), \quad \text{with } y(0) = y'(0) = 0. \quad (4.3)$$

Use the Laplace transform to determine the solution to this initial value problem and sketch the behaviour of the solution for all  $t \geq 0$ .

**4.4.4. The big picture.** Since we can also express any second or higher order ODEs as a system of first order equations, a nice way to visualize the process of solving linear constant coefficient ODE initial value problems via Laplace transform is as follows.



Hence instead of solving the corresponding system of first order ODEs with initial conditions directly (the arrow down the left-hand side), we solve the system indirectly, by first taking the Laplace transform of the ODEs and initial conditions—the arrow across the top. We must then solve the resulting linear algebraic problem for the Laplace transform of the solution  $\bar{\mathbf{y}}(s)$ . This corresponds to the arrow down the right-hand side. Then finally, to find the actual solution to the original initial value problem  $\mathbf{y}(t)$ , we must take the inverse Laplace transform—the arrow along the bottom.

## Table of Laplace transforms

| $f(t)$  | $\int_0^{\infty} e^{-st} f(t) dt$ |
|---|-----------------------------------|
| 1   | $\frac{1}{s}$                     |
| $t^n$   | $\frac{n!}{s^{n+1}}$              |
| $e^{at}$  | $\frac{1}{s-a}$                   |
| $\sin at$   | $\frac{a}{s^2 + a^2}$             |
| $\cos at$   | $\frac{s}{s^2 + a^2}$             |
| $af(t) + bg(t)$   | $a\bar{f}(s) + b\bar{g}(s)$       |
| $f'(t)$   | $s\bar{f}(s) - f(0)$              |
| $f''(t)$  | $s^2\bar{f}(s) - sf(0) - f'(0)$   |
| $e^{-at}f(t)$   | $\bar{f}(s+a)$                    |
| $f(t) = \begin{cases} g(t-a), & t \geq a, \\ 0, & t < a, \end{cases}$ | $e^{-sa}\bar{g}(s)$               |
| $\delta(t-a)$   | $e^{-sa}$                         |
| $f(t) * g(t)$   | $f(s) \cdot g(s)$                 |

TABLE 4.1. Table of Laplace transforms.



## Linear algebraic equations

### 5.1. Physical and engineering applications

When modelling many physical and engineering problems, we are often left with a system of algebraic equations for unknown quantities  $x_1, x_2, x_3, \dots, x_n$ , say. These unknown quantities may represent components of modes of oscillation in structures for example, or more generally<sup>1</sup>:

**Structures:** stresses and moments in complicated structures;

**Hydraulic networks:** hydraulic head at junctions and the rate of flow (discharge) for connecting pipes;

**Circuits:** electrical currents in circuits;

**Surveying:** error adjustments via least squares method;

**Curve fitting:** determining the coefficients of the best polynomial approximation;

**General networks:** abstract network problems—global communication systems;

**Finite difference schemes:** nodal values in the numerical implementation of a finite-difference scheme for solving differential equation boundary value problems;

**Finite element method:** elemental values in the numerical implementation of a finite element method for solving boundary value problems—useful in arbitrary geometrical configurations.

**Nonlinear cable analysis:** bridges, structures.

**Production totals:** factories, companies.

**Airline scheduling:** flight/ticket availability.

In any of these contexts, the system of algebraic equations that we must solve will in many cases be linear or at least can be well approximated by a linear system of equations. Linear algebraic equations are characterized by the property that no variable is raised to a power other than one or is multiplied by any other variable. The question is: is there a systematic procedure for solving such systems?

### 5.2. Systems of linear algebraic equations

**5.2.1. One equation with one unknown.** For example, suppose we are asked to solve:

$$2x = 6, \quad \text{clearly the solution is } x = 3.$$

*In general* if  $a$  and  $b$  are any two given real numbers, we might be asked to find the values of  $x$  for which

$$ax = b \tag{5.1}$$

is satisfied. There are three cases—we will return to this problem (5.1) time and again.

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<sup>1</sup>see [www.nr.com](http://www.nr.com) and [www.ulib.org](http://www.ulib.org)

- If  $a \neq 0$  then we can divide (5.1) by  $a$ . Thus  $x = b/a$  and this is the *unique solution*, i.e. the only value of  $x$  for which (5.1) is satisfied.
- If  $a = 0$  and  $b \neq 0$ , then there is *no solution*. There do not exist any real values of  $x$  such that  $0 \cdot x = b$  ( $\neq 0$ ).
- If  $a = 0$  and  $b = 0$ , then any value of  $x$  will satisfy  $0 \cdot x = 0$ . Hence there are *infinitely many solutions*.

**5.2.2. A system of two equations with two unknowns.** A simple example is the system of two equations for the unknowns  $x_1$  and  $x_2$ :

$$3x_1 + 4x_2 = 2, \quad (5.2)$$

$$x_1 + 2x_2 = 0. \quad (5.3)$$

There are many ways to solve this simple system of equations—we describe one that is easily generalised to much larger systems of linear equations.

*Step 1.* The first step is to eliminate  $x_1$  from (5.3) by replacing (5.3) by  $3 \cdot (5.3) - (5.2)$ :

$$3x_1 + 4x_2 = 2 \quad (5.4)$$

$$2x_2 = -2. \quad (5.5)$$

*Step 2.* Equation (5.5) is easy to solve (divide both sides by 2), and gives

$$x_2 = -1.$$

Substitution of this result back into (5.4) then gives

$$3x_1 - 4 = 2 \Rightarrow 3x_1 = 6 \Rightarrow x_1 = 2.$$

The solution to equations (5.2) and (5.3) is therefore  $x_1 = 2$  and  $x_2 = -1$ . The process in step 2 is known as *back-substitution*.

Note that having found the solution, we can always check it is correct by substituting our solution values into the equations, and showing that the equations are satisfied by these values.

$$3 \cdot (2) + 4 \cdot (-1) = 2$$

$$(2) + 2 \cdot (-1) = 0.$$

*Remark (algebraic interpretation).* What we have just done in the steps above is exactly equivalent to the following: first solve (5.2) for  $3x_1$ , i.e.  $3x_1 = 2 - 4x_2$ . Now multiply (5.3) by 3 so that it becomes  $3x_1 + 6x_2 = 0$ . Now substitute the expression for  $3x_1$  in the first equation into the second (thus eliminating  $x_1$ ) to get  $(2 - 4x_2) + 6x_2 = 0 \Rightarrow 2x_2 = -2$ , etc. . . .

*Remark (geometric interpretation).* The pair of simultaneous equations (5.2) & (5.3) also represent a pair of straight lines in the  $(x_1, x_2)$ -plane, rearranging:  $x_2 = -\frac{3}{4}x_1 + \frac{2}{3}$ ,  $x_2 = -\frac{1}{2}x_1$ . In posing the problem of finding the solution of this pair of simultaneous equations, we are asked to find the values of  $x_1$  and  $x_2$  such that both these constraints (each of these equations represents a constraint on the set of values of  $x_1$  and  $x_2$  in the plane) are satisfied simultaneously. This happens at the intersection of the two lines.

In general, consider the system of two linear equations:

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad (5.6)$$

$$a_{21}x_1 + a_{22}x_2 = b_2. \quad (5.7)$$

Solving these equations gives us

$$(5.6) \Rightarrow a_{11}x_1 + a_{12}x_2 = b_1, \quad (5.8)$$

$$a_{11} \cdot (5.7) - a_{21} \cdot (5.6) \Rightarrow \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_D x_2 = \underbrace{a_{11}b_2 - a_{21}b_1}_B. \quad (5.9)$$

For the system (5.6), (5.7) if

- $D \neq 0$ , then we may solve (5.9) to get  $x_2$ , and then by substituting this value back into (5.8) we determine  $x_1$ , i.e. there is a *unique solution* given by

$$x_2 = \frac{B}{D} = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}} \quad \text{and} \quad x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}.$$

- $D = 0$  and  $B \neq 0$  then there is *no solution*;
- $D = 0$  and  $B = 0$ , then any value of  $x_2$  will satisfy (5.9) and there is an *infinite number of solutions*.

**5.2.3. Remark.** The quantity  $D = a_{11}a_{22} - a_{12}a_{21}$  is clearly important, and is called the *determinant* of the system (5.6), (5.7). It is denoted by

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{or} \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}. \quad (5.10)$$

**5.2.4. A system of three equations with three unknowns.** A similar procedure can also be used to solve a system of three linear equations for three unknowns  $x_1, x_2, x_3$ . For example, suppose we wish to solve

$$2x_1 + 3x_2 - x_3 = 5, \quad (5.11)$$

$$4x_1 + 4x_2 - 3x_3 = 3, \quad (5.12)$$

$$2x_1 - 3x_2 + x_3 = -1. \quad (5.13)$$

First we eliminate  $x_1$  from equations (5.12) and (5.13) by subtracting multiples of (5.11). We replace (5.12) by (5.12)–2·(5.11) and (5.13) by (5.13)–(5.11), to leave the system

$$2x_1 + 3x_2 - x_3 = 5, \quad (5.14)$$

$$-2x_2 - x_3 = -7, \quad (5.15)$$

$$-6x_2 + 2x_3 = -6. \quad (5.16)$$

Next we eliminate  $x_2$  from (5.16). We do this by subtracting an appropriate multiple of (5.15). (If we subtract a multiple of (5.14) from (5.16) instead, then in the process of eliminating  $x_2$  from (5.16) we re-introduce  $x_1$ !). We therefore replace (5.16) by (5.16)–3·(5.15) to leave

$$2x_1 + 3x_2 - x_3 = 5, \quad (5.17)$$

$$-2x_2 - x_3 = -7, \quad (5.18)$$

$$5x_3 = 15. \quad (5.19)$$

Now that we have *triangularized* the system, we can use *back substitution* to find the solution: first solve (5.19) to give

$$x_3 = \frac{15}{5} = 3.$$

Then substitute this result back into (5.18) to give

$$-2x_2 - 3 = -7 \Rightarrow 2x_2 = 4 \Rightarrow x_2 = 2.$$

Finally we substitute the values of  $x_2$  and  $x_3$  back into (5.17) to give

$$2x_1 + 6 - 3 = 5 \Rightarrow 2x_1 = 2 \Rightarrow x_1 = 1.$$

So the solution to the system (5.11), (5.12), (5.13), is  $x_1 = 1$ ,  $x_2 = 2$  and  $x_3 = 3$ .

*Remark (algebraic interpretation).* Effectively, what we did in the steps above was to solve (5.11) for  $2x_1$ , multiply the result by 2, and substitute the resulting expression for  $4x_1$  into (5.12), thus eliminating  $x_1$  from that equation, and then also we have substituted our expression for  $2x_1$  from the first equation into (5.13) to eliminate  $x_1$  from that equation also, etc. . . .

*Remark (geometric interpretation).* An equation of the form  $ax + by + cz = d$ , where  $a$ ,  $b$ ,  $c$  and  $d$  are given constants, actually represents an infinite plane in three dimensional space. Thus (5.11), (5.12) and (5.13) represent three planes and we are asked to find the values of  $(x_1, x_2, x_3)$  such that each of the three equations is satisfied simultaneously—i.e. the point(s) of intersection of the three planes. If no two of the three planes are parallel (beware of the ‘toblerone’ case) then since two planes intersect in a line, and the third plane must cut that line at a single/unique point, we therefore have a unique solution in this case (where all three planes meet) which we deduce algebraically as above is  $x_1 = 1$ ,  $x_2 = 2$  and  $x_3 = 3$ .

### 5.3. Gaussian elimination

**5.3.1. The idea.** When modelling many physical and engineering problems we often need to simultaneously solve a large number of linear equations involving a large number of unknowns—there may be thousands of equations. We therefore need a systematic way of writing down (coding) and solving (processing) large systems of linear equations—preferably on a computer.

The method we used in the last section is known as *Gaussian elimination*, and can be applied to any size of system of equations. Just as importantly though, you may have noticed that to solve the system of linear equations by this method, we were simply performing operations on the coefficients in the equations, and the end result was an equivalent (triangularized) system of equations that was extremely easy to solve by back-substitution. With this in mind, a convenient way to write and analyse such systems is to use *matrices* as follows.

**5.3.2. Example.** To solve the system

$$\begin{aligned} 2x + 3y - z &= 5, \\ 4x + 4y - 3z &= 3, \\ 2x - 3y + z &= -1, \end{aligned}$$

we begin by writing down the *augmented matrix* of coefficients and right-hand sides:

$$H \equiv \left( \begin{array}{cccc} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ 2 & -3 & 1 & -1 \end{array} \right).$$

⏟
⏟  
 coeffs on LHS    RHS

Then following the Gaussian elimination procedure, we try to make the terms below the leading diagonal zero. Note that we do not need to write down the equations for  $x$ ,  $y$  &  $z$

at this stage, we can simply deal with the numerical coefficients in the augmented matrix  $H$ . The *advantage* of this approach using matrices, as we shall see, is that it is very easy to automate the solution process and implement the Gaussian elimination algorithm on a computer.

*Step 1: Clear the first column below the diagonal.* Replace Row2 by Row2 – 2·Row1:

$$\begin{pmatrix} \textcircled{2} & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 2 & -3 & 1 & -1 \end{pmatrix}$$

And now replace Row3 by Row3 – Row1:

$$\begin{pmatrix} \textcircled{2} & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & -6 & 2 & -6 \end{pmatrix}$$

*Step 2: Clear the second column below the diagonal.* Replace Row3 by Row3 – 3·Row2:

$$\begin{pmatrix} 2 & 3 & -1 & 5 \\ 0 & \textcircled{-2} & -1 & -7 \\ 0 & 0 & 5 & 15 \end{pmatrix}$$

*Step 3: Use back-substitution to find the solution.* Now we rewrite the rows of the augmented matrix as equations for  $x$ ,  $y$  &  $z$  and proceed to solve the system of equations by *back-substitution* (a process which can also be coded and automated) giving us the solution we had before:

$$\begin{aligned} \text{Row 3} &\Rightarrow 5z = 15 \quad \Rightarrow \quad z = 3; \\ \text{Row 2} &\Rightarrow -2y - z = -7 \quad \Rightarrow \quad y = 2; \\ \text{Row 1} &\Rightarrow 2x - 3y + z = 5 \quad \Rightarrow \quad x = 1. \end{aligned}$$

**5.3.3. Pivots.** Note that in the first two steps we focused on the *pivot position*—the encircled numbers—which we use to eliminate all the terms below that position. The coefficient in the pivot position is called the *pivotal element* or *pivot*. The pivotal element must always be non-zero—if it is ever zero, then the pivotal row/equation is interchanged with an equation below it to produce a non-zero pivot—see the next example. This is always possible for systems with a unique solution. When implementing the Gaussian elimination algorithm on a computer, to minimize rounding errors, a practice called *partial pivoting* is used, whereby we interchange (if necessary) the pivot row with any row below it, to ensure that the pivot has the maximum possible magnitude—we will discuss partial pivoting in much more detail shortly—also see Meyer [10].

**5.3.4. Example (of simple pivoting).** The following system of equations models the currents in the electrical circuit—see system (A.2) in Appendix D.

$$\begin{aligned} I_1 - I_2 + I_3 &= 0, \\ -I_1 + I_2 - I_3 &= 0, \\ 10I_2 + 25I_3 &= 90, \\ 20I_1 + 10I_2 &= 80. \end{aligned}$$





Now use back-substitution to obtain the solution (starting from the bottom):

$$\begin{aligned}\text{Row 4} &\Rightarrow 2z = 4 \Rightarrow z = 2; \\ \text{Row 3} &\Rightarrow 3y - 2z = 7 \Rightarrow y = 3; \\ \text{Row 2} &\Rightarrow 6x - 3y + 2z = 1 \Rightarrow x = 1; \\ \text{Row 1} &\Rightarrow 2w + 4x + y + 2z = 5 \Rightarrow w = -3.\end{aligned}$$

**5.3.8. Definition (determined systems).** Every system of linear equations can be classified as follows.

- When the number of equations equals the number of unknowns the system is said to be *determined*.
- When the number of equations is less than the number of unknowns the system is said to be *underdetermined*.
- When the number of equations is greater than the number of unknowns the system is said to be *overdetermined*.

**5.3.9. Example.** Consider the system

$$\begin{aligned}3x + 2y + z &= 3, \\ 2x + y + z &= 0, \\ 6x + 2y + 4z &= 6.\end{aligned}$$

This system is determined. But does it have a solution? We can attempt to find the solution by constructing the augmented matrix and performing EROs.

$$H \equiv \begin{pmatrix} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{pmatrix}$$

$$\begin{aligned}R2 &\rightarrow R2 - \frac{2}{3}R1; \\ R3 &\rightarrow R3 - 2R1 : \end{aligned} \quad \begin{pmatrix} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & -2 & 2 & 0 \end{pmatrix}$$

$$R3 \rightarrow R3 - 6R2 : \quad \begin{pmatrix} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{pmatrix}$$

i.e.

$$\begin{aligned}3x + 2y + z &= 3, \\ -\frac{1}{3}y + \frac{1}{3}z &= 2, \\ 0z &= 12.\end{aligned}$$

This system has no solution (there is no solution to the last equation,  $0 \cdot z = 12$ ).

When the system is *underdetermined*, a solution can *always* be found, though it will not be unique—for example when we have two equations for three unknowns, the solution is the intersection of two planes, which is a line, and there are infinitely many solutions. When the system is either *determined* or *overdetermined*, solutions may or may not exist.

### 5.4. Matrix Equations

**5.4.1. Example.** The system of equations

$$2x_1 + 3x_2 - x_3 = 5, \quad (5.22a)$$

$$4x_1 + 4x_2 - 3x_3 = 3, \quad (5.22b)$$

$$2x_1 - 3x_2 + x_3 = -1, \quad (5.22c)$$

we solved previously may be written more compactly by introducing

$$A = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 1 \end{pmatrix} = \text{the matrix of coefficients,}$$

with

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \text{the vector of unknowns}$$

and

$$\mathbf{b} = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} = \text{the vector of right-hand sides.}$$

Then we simply write the system of equations (5.22) as

$$A\mathbf{x} = \mathbf{b}. \quad (5.23)$$

**5.4.2. General linear systems.** For large systems of equations the notation (5.23) is very compact.

*More generally*, if we have a system of  $m$  equations in  $n$  unknowns

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned} \right\} \quad (5.24)$$

these may be written in matrix form as

$$A\mathbf{x} = \mathbf{b}, \quad (5.25)$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Here  $A$  is an  $m \times n$  matrix ( $m$  = number of rows,  $n$  = number of columns),  $\mathbf{x}$  is an  $n \times 1$  matrix (also known as a column vector),  $\mathbf{b}$  is an  $m \times 1$  matrix (again, a column vector). Note that  $a_{ij}$  = element of  $A$  at the intersection of the  $i$ -th row and the  $j$ -th column. The dimensions or size, ' $m \times n$ ', of the matrix  $A$ , is also known as the *order* of  $A$ .

**5.4.3. Multiplication of matrices.** Multiplication of one matrix by another is a more involved operation than you might expect. To motivate matrix multiplication, recall that we rewrote the system (5.24) in the form  $A\mathbf{x} = \mathbf{b}$ , i.e.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

We therefore want the product  $A\mathbf{x}$  to mean

$$A\mathbf{x} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ a_{21}x_1 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix}. \quad (5.26)$$

To make sense of matrix multiplication we need: number of columns of  $A$  = number of rows of  $\mathbf{x}$ . Thus

$$\underbrace{\begin{matrix} A & \mathbf{x} \\ (m \times n) & (n \times 1) \\ \text{equal} \\ m \times 1 \text{ matrix} \end{matrix}} = \mathbf{b} \quad (m \times 1)$$

Matrix multiplication of an  $m \times n$  matrix  $A$  by an  $n \times 1$  matrix  $\mathbf{x}$  is defined by (5.26).

## 5.5. Linear independence

**5.5.1. Definition (linear combination).** Given any set of  $m$  vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$  (with the same number of components in each), a *linear combination* of these vectors is an expression of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$$

where  $c_1, c_2, \dots, c_m$  are any scalars.

**5.5.2. Definition (linear independence).** Now consider the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = \mathbf{0}. \quad (5.27)$$

- The vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$  are said to be *linearly independent* if the only values for  $c_1, c_2, \dots, c_m$  for which (5.27) holds are when  $c_1, c_2, \dots, c_m$  are all zero, i.e.

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = \mathbf{0} \quad \Rightarrow \quad c_1 = c_2 = \cdots = c_m = 0.$$

- If (5.27) holds for some set of  $c_1, c_2, \dots, c_m$  which are not all zero, then the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$  are said to be *linearly dependent*, i.e. we can express at least one of them as a linear combination of the others. For instance, if (5.27) holds with, say  $c_1 \neq 0$ , we can solve (5.27) for  $\mathbf{v}_1$ :

$$\mathbf{v}_1 = k_2\mathbf{v}_2 + \cdots + k_m\mathbf{v}_m \quad \text{where} \quad k_i = -\frac{c_i}{c_1}.$$

**5.5.3. Example.** The three vectors

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -6 \\ 42 \\ 24 \\ 54 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 21 \\ -21 \\ 0 \\ -15 \end{pmatrix}$$

are linearly dependent because

$$6\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}.$$

However note that  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent because

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0},$$

implies  $c_2 = 0$  (from the second components) and then  $c_1 = 0$  (from any of the other components).

Although it is easy to verify that  $\mathbf{v}_1$ ,  $\mathbf{v}_2$  and  $\mathbf{v}_3$  are linearly dependent given an appropriate linear combination, it is not so easy to determine if a given set of vectors is either linearly dependent or independent when starting from scratch. Next we show a method to help us in this direction.

## 5.6. Rank of a matrix

**5.6.1. Definition (rank).** The maximum number of linearly independent row vectors of a matrix  $A = [a_{ij}]$  is called the *rank* of  $A$  and is denoted by ‘rank[ $A$ ]’.

**5.6.2. Example.** The matrix

$$A = \begin{pmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{pmatrix}$$

has rank 2, since from the previous example above, the first two row vectors are linearly independent, whereas the three row vectors are linearly dependent.

Note further that,  $\text{rank}[A] = 0 \Leftrightarrow A = O$ . This follows directly from the definition of rank. Another surprising result is that *row-rank equals column-rank*, i.e. the rank of a matrix  $A$  equals the maximum number of linearly independent column vectors of  $A$ , or equivalently, the matrix  $A$  and its transpose  $A^T$ , have the same rank<sup>2</sup>. However, the most important result of this section concerns *row-equivalent matrices*—these are matrices that can be obtained from each other by finitely many EROs.

- *Invariance of rank under EROs.* Row-equivalent matrices have the same rank, i.e. EROs do not alter the rank of a matrix  $A$ .
- *Test for linear dependence/independence.* A given set of  $p$  vectors,  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$  (with  $n$  components each), are linearly independent if the matrix with row vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$  have rank  $p$ ; they are linearly dependent if that rank is less than  $p$ .

Hence to determine the rank of a matrix  $A$ , we can reduce  $A$  to echelon form using Gaussian elimination (this leaves the rank unchanged) and from the echelon form we can recognize the rank directly. Further, the second result above implies that, to test if a given set of vectors is linearly dependent/independent, we combine them to form rows (or columns) of a matrix and reduce that matrix to echelon form using Gaussian elimination. If the rank of

<sup>2</sup>See the end of appendix C for the definition of the transpose of a matrix.

the resulting matrix is equal to the number of given vectors then they're linearly independent, otherwise, they're linearly dependent.

### 5.6.3. Example.

$$A = \begin{pmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{pmatrix}$$

$R2 \rightarrow R2 + 2R1$  and  $R3 \rightarrow R3 - 7R1$  implies

$$\begin{pmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{pmatrix}$$

$R3 \rightarrow R3 + \frac{1}{2}R2$  implies

$$\begin{pmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where the matrix is now in echelon form. Hence  $\text{rank}[A] = 2$ .

## 5.7. Solution of general rectangular systems

Suppose we are required to solve the system of  $m$  linear equations in  $n$  unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

The nature of the solution is easily determined if we use elementary row operations (EROs) to reduce the augmented matrix  $H = [A \ \mathbf{b}']$  to *row-echelon* form—see Figure 5.7.

There are three cases (referring to Figure 5.7 directly):

- a *unique solution* if  $r = n$  and  $b'_{r+1}, \dots, b'_m$  are all zero; and the solution can be obtained by back-substitution;
- an *infinite number of solutions* if  $r < n$  and  $b'_{r+1}, \dots, b'_m$  are all zero;
- *no solution* if  $r < m$  and one of  $b'_{r+1}, \dots, b'_m$  is non-zero.

The system of equations is said to be *consistent* if there is at least one solution (either a unique solution or an infinite number of solutions) and *not consistent* if there is no solution. We can also characterise the different possibilities using the notion of *rank* as follows—the fact that EROs do not alter the rank of a matrix establishes the equivalence of both statements.

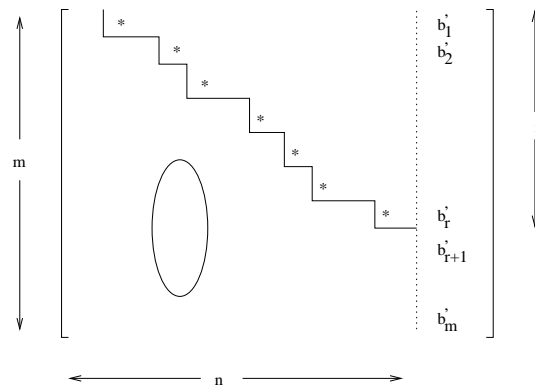


FIGURE 5.1.  $[A \mathbf{b}]$  reduced to row-echelon form ( $r \leq \min\{m, n\}$ ); the \*'s represent non-zero elements—the other elements above the diagonal line may or may not be zero.

*Fundamental theorem for linear systems.* A linear system of  $m$  equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1,$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2,$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m,$$

in  $n$  unknowns  $x_1, x_2, \dots, x_n$  has solutions if and only if the coefficient matrix  $A$  and the augmented matrix  $H = [A \mathbf{b}]$  have the same rank, i.e. if and only if

$$\text{rank}[H] = \text{rank}[A].$$

- If this rank,  $r$ , equals  $n$ , then the system has a *unique solution*.
- If  $r < n$ , the system has *infinitely many solutions* (all of which are obtained by determining  $r$  suitable unknowns in terms of the remaining  $n - r$  unknowns, to which arbitrary values can be assigned).

If solutions exist, they can be obtained by Gaussian elimination.



## Gaussian elimination in practice

### 6.1. Gauss-Jordan method

**6.1.1. Example.** Another method for solving linear systems of algebraic equations is the *Gauss-Jordan method*. It is a continuation of the Gaussian elimination process.

$$\begin{aligned} 3x - y + 2z &= 3, \\ 2x + y + z &= -2, \\ x - 3y &= 5. \end{aligned}$$

Start by constructing the augmented matrix:

$$H = \begin{pmatrix} 3 & -1 & 2 & 3 \\ 2 & 1 & 1 & -2 \\ 1 & -3 & 0 & 5 \end{pmatrix}.$$

$$\begin{aligned} R2 &\rightarrow 3R2 - 2R1; \\ R3 &\rightarrow 3R3 - R1; \end{aligned} \quad \begin{pmatrix} 3 & -1 & 2 & 3 \\ 0 & 5 & -1 & -12 \\ 0 & -8 & -2 & 12 \end{pmatrix}$$

$$R3 \rightarrow 5R3 + 8R2; \quad \begin{pmatrix} 3 & -1 & 2 & 3 \\ 0 & 5 & -1 & -12 \\ 0 & 0 & -18 & -36 \end{pmatrix}.$$

At this point, with Gaussian elimination, we would back-substitute by first solving the 3rd equation etc. In the Gauss-Jordan method we continue applying EROs to reduce the left-hand submatrix to the  $3 \times 3$  identity matrix:

$$\begin{aligned} R1 &\rightarrow R1 + \frac{1}{9}R3; \\ R2 &\rightarrow 2R2 - \frac{1}{9}R3; \end{aligned} \quad \begin{pmatrix} 3 & -1 & 0 & -1 \\ 0 & 10 & 0 & -20 \\ 0 & 0 & -18 & -36 \end{pmatrix}$$

$$R1 \rightarrow R1 + \frac{1}{10}R2; \quad \begin{pmatrix} 3 & 0 & 0 & -3 \\ 0 & 10 & 0 & -20 \\ 0 & 0 & -18 & -36 \end{pmatrix}$$

$$\begin{aligned} R1 &\rightarrow \frac{1}{3}R1; \\ R2 &\rightarrow \frac{1}{10}R2; \\ R3 &\rightarrow -\frac{1}{18}R3; \end{aligned} \quad \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

The solution to this system is therefore  $x = -1$ ,  $y = -2$  and  $z = 2$ .

### 6.2. Operation counts

Which method should we use? Gaussian elimination or the Gauss-Jordan method? The answer lies in the *efficiency* of the respective methods when solving large systems.

*Gaussian elimination.* Gaussian elimination with back-substitution applied to an  $n \times n$  system requires

$$\left. \begin{array}{l} \frac{n^3}{3} + n^2 - \frac{n}{3} \text{ multiplications/divisions} \\ \frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6} \text{ additions/subtractions} \end{array} \right\} \Rightarrow \frac{2n^3}{3} + \frac{3n^2}{2} - \frac{7n}{6} \text{ flops.}$$

For large values of  $n$ , i.e. large systems, the  $2n^3/3$  term dominates and we would normally quote that Gaussian elimination with back substitution for an  $n \times n$  system requires  $\sim 2n^3/3$  flops (flops  $\equiv$  floating point operations).

*Gaussian-Jordan.* Gaussian-Jordan applied to an  $n \times n$  system requires

$$\left. \begin{array}{l} \frac{n^3}{2} + \frac{n^2}{2} \text{ multiplications/divisions} \\ \frac{n^3}{2} - \frac{n}{2} \text{ additions/subtractions} \end{array} \right\} \Rightarrow n^3 + \frac{n^2}{2} - \frac{n}{2} \text{ flops.}$$

For large values of  $n$  Gaussian-Jordan requires  $\sim n^3$  flops.

Hence Gauss-Jordan requires about 50% more effort than Gaussian elimination—and this difference becomes significant when  $n$  is large. For example if  $n = 100$ , then  $2n^3/3 \approx 666,666$  while  $n^3 = 1,000,000$  which amounts to 333,333 extra flops—see Meyer [10].

Thus Gauss-Jordan is not recommended for solving linear systems of equations that arise in practical situations, though it does have theoretical advantages—for example for finding the inverse of a matrix.

### 6.3. Partial pivoting and scaling

**6.3.1. Finite accuracy computer arithmetic.** When implementing Gaussian elimination on a computer, in practice for large systems of equations, we need to contend with the fact that we can only perform calculations to a given finite accuracy (modern algebraic computing packages can perform exact integer arithmetic but become impractical for large systems).

**6.3.2. Example.** First find the exact solution, and then use three digit accuracy arithmetic to find an approximate solution of the following system:

$$\begin{aligned} 47x + 28y &= 19, \\ 89x + 53y &= 36. \end{aligned}$$

**6.3.3. Solution.** The exact solution is

$$x = 1 \quad \text{and} \quad y = -1.$$

However using three digit arithmetic you get

$$x = -0.191 \quad \text{and} \quad y = 1.00.$$

**6.3.4. Partial pivoting.** In this last example, rounding errors are responsible for the anomalous answer, and though such errors can never be completely eliminated, there are some practical strategies to help minimize these errors.

*Partial pivoting.* The idea is to maximize the magnitude of the pivot at each step using only row interchanges. More precisely, at each step we interchange (if necessary) the pivot row with any row below it to ensure that the pivot has the maximum possible magnitude compared with the coefficients in the column below its position. The fourth step in a typical case would be:

$$\left( \begin{array}{cccccc|c} * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & \textcircled{P} & * & * & * \\ 0 & 0 & 0 & P & * & * & * \\ 0 & 0 & 0 & P & * & * & * \end{array} \right)$$

Search the positions in the fourth column and then interchange the rows so that the coefficient in the circled position is the largest in magnitude of all the terms labelled “P”.

**6.3.5. Example.** Find the exact solution, then the approximate solution using three digit accuracy, first without using partial pivoting, and then subsequently using partial pivoting:

$$\begin{aligned} -10^{-4}x + y &= 1, \\ x + y &= 2. \end{aligned}$$

*Note.* When partial pivoting is used, no multiplier ever exceeds 1 in magnitude—and the possibility that large numbers can “swamp” small ones in finite digit arithmetic is significantly reduced but not entirely eliminated.

**6.3.6. Example (two-scale system).** Find the exact solution, then the approximate solution using three digit accuracy and partial pivoting of the system of equations:

$$\begin{aligned} -10x + 10^5y &= 10^5, \\ x + y &= 2. \end{aligned}$$

**6.3.7. Scaling.** We saw in this last example that the differing orders of magnitude of the coefficients means that the first equation applies on a different *scale* to the second, and this produced the anomalous approximate solution. A sensible strategy would be to rescale the system before we proceed to solve it.

*The idea* is to rescale the above system so that the coefficient of maximum possible magnitude is 1. For example if we multiply the first equation in the last example by  $10^{-5}$ , we recover the example system we saw in the partial pivoting section, which we were very adequately able to solve using partial pivoting.

Hence the second refinement need for successful Gaussian elimination is a scaling strategy: which in fact combines both *row scaling*—multiplying selected rows by non-zero multipliers— with *column scaling*—multiplying selected columns of the coefficient matrix  $A$  by non-zero multipliers.

*Note.* Row scaling does not change the exact solution, but column scaling does! Column scaling the  $k$ th column is equivalent to changing the units of the  $k$ th unknown.

*Practical scaling strategy.*

- Always choose units that are natural to the problem so that there is not a distorted relationship between the sizes of physical parameters within a given problem.
- Row scale the system of linear equations so that the maximum magnitude of the coefficient in each row is 1, i.e. divide each row by the coefficient (in that row) of maximum magnitude.

Partial pivoting and this scaling strategy makes Gaussian elimination with back substitution a proven extremely reliable and effective tool for practical systems of linear equations.

#### 6.4. Ill-conditioned systems

**6.4.1. Example.** This example demonstrates that there are some systems that are very sensitive to small perturbations and why care must always be taken when numerically solving equations. Consider the system

$$.835x + .667y = .168, \quad (6.1a)$$

$$.333x + .266y = b_2. \quad (6.1b)$$

First let's suppose that  $b_2 = .067$ , then the exact solution is

$$x = 1. \quad \text{and} \quad y = -1. \quad (\text{exact values}).$$

Let's perturb  $b_2$  slightly to  $b_2 = .066$ . Now the exact solution is

$$x = -666. \quad \text{and} \quad y = 834. \quad (\text{exact values}).$$

This is an example of an ill-conditioned system. It's sensitivity to small perturbations is intrinsic to the system itself, rather than the result of any numerical procedure.

*Ill-conditioned systems:* A system of equations is said to be *ill-conditioned* when some small perturbation in the system can produce relatively large changes in the exact solution. Otherwise the system is said to be *well-conditioned*.

**6.4.2. Geometric interpretation.** Let's examine what's happening here from a geometric perspective. The equations (6.1) above represent two straight lines (with  $b_2 = .067$ ):

$$y = -1.25187x + .251874, \quad (6.2a)$$

$$y = -1.25188x + .25188. \quad (6.2b)$$

Notice that these two lines have very similar slopes and intercepts—they are extremely close to each other—see Figure 6.4.2. They do intersect at the point  $x = 1.$  and  $y = -1.$  when  $b_2 = .067.$  However it is easy to imagine how if we were to perturb one of the lines slightly, the point of intersection (solution) could change quite dramatically (as we have seen).

#### 6.5. Matrix Inversion

**6.5.1. Inversion via EROs.** Row reduction methods can be used to find the inverse of a matrix—in particular via the Gauss-Jordan approach.

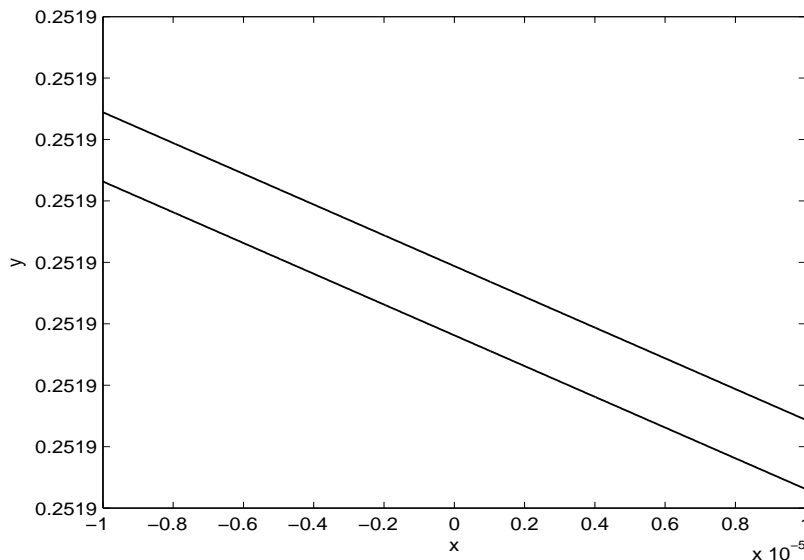


FIGURE 6.1. The two lines  $.835x + .667y = .168$  and  $.333x + .266y = .067$  are extremely close in and around the origin  $x = 0$ ,  $y = 0$ —relatively far from the root  $x = 1$ ,  $y = -1$ .

**6.5.2. Example.** Calculate the inverse of  $A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix}$ .

**6.5.3. Solution.** Consider the augmented matrix

$$\left( \begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 5 & 0 & 0 & 1 \end{array} \right).$$

matrix  $A$                   identity

The plan is to perform row operations on this augmented matrix in such a way as to reduce it to the form

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{array} \right).$$

identity                  matrix  $B$

The  $3 \times 3$  matrix  $B$  is then guaranteed to be  $A^{-1}$ , the inverse of  $A$ .

$$\begin{array}{l} R_2 \rightarrow R_2 - 2R_1; \\ R_3 \rightarrow R_3 - R_1; \end{array} \quad \left( \begin{array}{ccc|ccc} 1 & 1 & 3 & \vdots & 1 & 0 & 0 \\ 0 & -1 & -5 & \vdots & -2 & 1 & 0 \\ 0 & 2 & 2 & \vdots & -1 & 0 & 1 \end{array} \right)$$

This pair of row operations clears the elements in column 1, rows 2 and 3.

$$R_3 \rightarrow R_3 + 2R_2; \quad \left( \begin{array}{ccc|ccc} 1 & 1 & 3 & \vdots & 1 & 0 & 0 \\ 0 & -1 & -5 & \vdots & -2 & 1 & 0 \\ 0 & 0 & -8 & \vdots & -5 & 2 & 1 \end{array} \right)$$

This clears column 2, row 3. Now we try to “clear” the top half:

$$\begin{aligned}
 R_2 &\rightarrow R_2 - \frac{5}{8}R_3; \\
 R_1 &\rightarrow R_1 + \frac{3}{8}R_3; \\
 \\
 R_1 &\rightarrow R_1 + R_2; \\
 \\
 R_2 &\rightarrow -R_2; \\
 R_3 &\rightarrow -\frac{1}{8}R_3; \\
 \\
 \Rightarrow A^{-1} &= \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{9}{8} & \frac{1}{4} & \frac{5}{8} \\ \frac{5}{8} & -\frac{1}{4} & -\frac{1}{8} \end{pmatrix}.
 \end{aligned}$$

Now check that  $AA^{-1} = I$ .

**6.5.4. In reality.** We very rarely need to invert matrices. However if we do need to do so on a computer, then when inverting a matrix, say  $A$ , it's roughly equivalent, in terms of raw numerical computations, to use the Gauss-Jordan procedure described above as opposed to using Gaussian elimination to perform EROs on augmented matrix  $[A: I]$  so that the left-hand matrix becomes upper diagonal, and then to use back-substitution on each of the three underlying problems to find each of the three columns of  $A^{-1}$ , see Meyer [10].

## Linear algebraic eigenvalue problems

### 7.1. Eigenvalues and eigenvectors

In this chapter we study eigenvalue problems. These arise in many situations, for example: calculating the natural frequencies of oscillation of a vibrating system; finding principal axes of stress and strain; calculating oscillations of an electrical circuit; image processing; data mining (web search engines); etc.

**Differential equations:** solving arbitrary order linear differential equations analytically;

**Vibration analysis:** calculating the natural frequencies of oscillation of a vibrating system—bridges, cantilevers;

**Principal axes of stress and strain:** mechanics;

**Dynamic stability:** linear stability analysis;

**Column buckling:** lateral deflections—modes of buckling;

**Electrical circuit:** oscillations, resonance;

**Principle component analysis:** extracting the salient features of a mass of data;

**Markov Chains:** transition matrices;

**Data mining:** web search engines—analysis of fixed point problems;

**Image processing:** fixed point problems again;

**Quantum mechanics:** quantized energy levels.

In fact when we solved the linear second order equations in Chapters 2–3, we were actually solving an eigenvalue problem.

*Eigenvalue problems* take the form

$$A\mathbf{x} = \lambda\mathbf{x}, \quad (7.1)$$

where  $A$  is a square  $n \times n$  matrix and  $\mathbf{x}$  is an  $n \times 1$  matrix (column vector).

- *The problem* is to find values of  $\lambda$  for which (7.1) has a *nontrivial* (nonzero) solution  $\mathbf{x}$ .
- Such values of  $\lambda$  are called *eigenvalues*, and the corresponding vectors  $\mathbf{x}$ , are the *eigenvectors* of the matrix  $A$ .

Equation (7.1) may be written as follows (here  $I$  is the identity matrix)

$$\begin{aligned} & A\mathbf{x} = \lambda\mathbf{x} \\ \Leftrightarrow & A\mathbf{x} = \lambda I\mathbf{x} \\ \Leftrightarrow & A\mathbf{x} - \lambda I\mathbf{x} = \mathbf{0} \\ \Leftrightarrow & (A - \lambda I)\mathbf{x} = \mathbf{0}. \end{aligned} \quad (7.2)$$

As  $\lambda$  is varied, the entries of the matrix  $A - \lambda I$  vary. If  $\det(A - \lambda I) \neq 0$  the matrix  $A - \lambda I$  can be inverted, and the only solution of (7.2) is

$$\mathbf{x} = (A - \lambda I)^{-1} \mathbf{0} = \mathbf{0}.$$

- Nonzero solutions to (7.2)  $\mathbf{x}$  will exist provided  $\lambda$  is such that

$$\det(A - \lambda I) = 0. \quad (7.3)$$

Equation (7.3) is called the *characteristic equation*.

- If  $A$  is an  $n \times n$  matrix then  $\det(A - \lambda I)$  is an  $n$ -th degree polynomial in  $\lambda$ , i.e. it has the form

$$\det(A - \lambda I) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0. \quad (7.4)$$

We call (7.4) the *characteristic polynomial*.

- Hence (7.3) is equivalent to

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0, \quad (7.5)$$

and the solutions of this equation  $\lambda_1, \dots, \lambda_n$  are *eigenvalues*.

By the *Fundamental Theorem of Algebra* there are  $n$  roots of a polynomial of degree  $n$  (including repetitions); hence there are  $n$  solutions to the characteristic equation (including repetitions).

Note also that if  $\mathbf{x}$  is an eigenvector of  $A$  with eigenvalue  $\lambda$ , and  $c \neq 0$  is a scalar then  $c\mathbf{x}$  is also an eigenvector of  $A$  with eigenvalue  $\lambda$ , since

$$A(c\mathbf{x}) = cA\mathbf{x} = c\lambda\mathbf{x} = \lambda(c\mathbf{x}).$$

This is simply due to the fact that (7.1) is a *homogeneous* algebraic system of equations.

**7.1.1. Example.** Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}.$$

**7.1.2. Solution.** First we solve the characteristic equation.

$$\det(A - \lambda I) = 0 \quad \Leftrightarrow \quad \det \begin{pmatrix} 3 - \lambda & -2 \\ 4 & -3 - \lambda \end{pmatrix} = 0 \quad \Leftrightarrow \quad \lambda = \pm 1.$$

For *each* eigenvalue we must find the corresponding eigenvector. Write  $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ .

*Case*  $\lambda = 1$ :

$$\begin{aligned} A\mathbf{x} = \lambda\mathbf{x} &\Rightarrow \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix}, \\ &\Leftrightarrow \begin{aligned} 3x - 2y = x &\Rightarrow x = y \\ 4x - 3y = y &\Rightarrow x = y. \end{aligned} \end{aligned}$$

So any vector of the form  $\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  is an eigenvector for any  $\alpha \neq 0$ .

Case  $\lambda = -1$ :

$$\begin{aligned} A\mathbf{x} = \lambda\mathbf{x} &\Rightarrow \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -1 \begin{pmatrix} x \\ y \end{pmatrix}, \\ &\Leftrightarrow \begin{aligned} 3x - 2y = -x &\Rightarrow 2x = y \\ 4x - 3y = -y &\Rightarrow 2x = y. \end{aligned} \end{aligned}$$

Hence any vector of the form  $\beta \begin{pmatrix} 1 \\ 2 \end{pmatrix}$  is an eigenvector for any  $\beta \neq 0$ .

**7.1.3. Example.** Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 4 & -2 & -4 \\ 1 & 5 & 1 \\ -2 & -2 & 2 \end{pmatrix}.$$

**7.1.4. Example.** Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}.$$

**7.1.5. Solution.** We first find the eigenvalues  $\lambda$  from the characteristic equation,

$$\det \begin{pmatrix} 2 - \lambda & -2 & 2 \\ 1 & 1 - \lambda & 1 \\ 1 & 3 & -1 - \lambda \end{pmatrix} = 0 \Leftrightarrow -(\lambda + 2)(\lambda - 2)^2 = 0,$$

which has the solutions 2 (twice) and  $-2$ . For each eigenvalue we must calculate the corresponding eigenvectors.

Case  $\lambda = -2$ :

$$\begin{aligned} A\mathbf{x} = \lambda\mathbf{x} &\Rightarrow \begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &\Leftrightarrow \begin{aligned} 2x - 2y + 2z &= -2x, \\ x + y + z &= -2y, \\ x + 3y - z &= -2z, \\ 2x - y + z &= 0, \\ x + 3y + z &= 0, \\ x + 3y + z &= 0. \end{aligned} \end{aligned}$$

Note that the second and third equations are equivalent, so one equation is redundant. We are left then with two equations for three unknowns. We use these to write  $y$  and  $z$  in terms of  $x$ .

$$\text{1st equation} \Rightarrow -y + z = -2x \tag{7.6}$$

$$\text{2nd equation} \Rightarrow 3y + z = -x \tag{7.7}$$

$$3 \cdot (7.6) + (7.7) \Rightarrow 4z = -7x \Rightarrow z = \frac{-7}{4}x$$

Substituting this expression for  $z$  into (7.6) gives  $y = z + 2x = \frac{1}{4}x$ . So the eigenvector corresponding to  $\lambda = -2$  is

$$x \begin{pmatrix} 1 \\ 1/4 \\ -7/4 \end{pmatrix}$$

for any  $x \neq 0$ .

Case  $\lambda = 2$ :

$$\begin{aligned} A\mathbf{x} = \lambda\mathbf{x} &\Rightarrow \begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &2x - 2y + 2z = 2x, \\ &\Leftrightarrow \quad x + y + z = 2y, \\ &\quad x + 3y - z = 2z, \\ &\quad -y + z = 0, \\ &\Leftrightarrow \quad x - y + z = 0, \\ &\quad x + 3y - 3z = 0. \end{aligned}$$

The first equation implies  $y = z$ . Then the second equation implies  $x = 0$ . The third equation is then consistent with these last two deductions. Hence the eigenvector corresponding to  $\lambda = 2$  is

$$y \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

for any  $y \neq 0$ . What about the (potentially) third eigenvector? There is not necessarily a third one, we'll come to this point in detail when we discuss multiplicity after the next example.

**7.1.6. Example.** Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$$

**7.1.7. Solution.** We first find the eigenvalues  $\lambda$  from the characteristic equation

$$\det \begin{pmatrix} -2 - \lambda & 2 & -3 \\ 2 & 1 - \lambda & -6 \\ -1 & -2 & -\lambda \end{pmatrix} = 0 \quad \Leftrightarrow \quad (\lambda - 5)(\lambda + 3)^2 = 0$$

which has the solutions  $\lambda = -3$  (twice) and  $\lambda = 5$ . For each eigenvalue we must calculate the corresponding eigenvectors.

Case  $\lambda = 5$ :

$$\begin{aligned} A\mathbf{x} = \lambda\mathbf{x} &\Rightarrow \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &-2x + 2y - 3z = 5x, \\ &\Leftrightarrow \quad 2x + y - 6z = 5y, \\ &\quad -x - 2y + 0z = 5z, \end{aligned}$$

$$\begin{aligned} & -7x + 2y - 3z = 0, \\ \Leftrightarrow & \quad 2x - 4y - 6z = 0, \\ & \quad -x - 2y - 5z = 0. \end{aligned}$$

We can now solve this system of linear equations by Gaussian elimination. Hence, consider the augmented matrix

$$\begin{aligned} H & \equiv \begin{pmatrix} -7 & 2 & -3 & 0 \\ 2 & -4 & -6 & 0 \\ -1 & -2 & -5 & 0 \end{pmatrix} \\ R2 & \rightarrow 7R2 + 2R1; & \begin{pmatrix} -7 & 2 & -3 & 0 \\ 0 & -24 & -48 & 0 \\ 0 & -16 & -32 & 0 \end{pmatrix} \\ R3 & \rightarrow 7R3 - R1; & \begin{pmatrix} -7 & 2 & -3 & 0 \\ 0 & -24 & -48 & 0 \\ 0 & -16 & -32 & 0 \end{pmatrix} \\ R3 & \rightarrow 3R3 - 2R2; & \begin{pmatrix} -7 & 2 & -3 & 0 \\ 0 & -24 & -48 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Hence we can solve these equations to get  $y = -2z$ ,  $x = -z$ . So the eigenvector corresponding to  $\lambda = 5$  is

$$\mathbf{v}_1 = \alpha \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

for any value  $z = \alpha$  ( $\neq 0$ ).

Case  $\lambda = -3$ :

$$\begin{aligned} A\mathbf{x} = \lambda\mathbf{x} & \Leftrightarrow \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -3 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ & \quad -2x + 2y - 3z = -3x, \\ \Leftrightarrow & \quad 2x + y - 6z = -3y, \\ & \quad -x - 2y + 0z = -3z, \\ & \quad x + 2y - 3z = 0, \\ \Leftrightarrow & \quad 2x + 4y - 6z = 0, \\ & \quad -x - 2y + 3z = 0. \end{aligned}$$

We can now solve this system of linear equations by Gaussian elimination. Hence, consider the augmented matrix

$$\begin{aligned} H & \equiv \begin{pmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ -1 & -2 & 3 & 0 \end{pmatrix} \\ R2 & \rightarrow R2 - 2R1; & \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ R3 & \rightarrow R3 + R1; & \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Hence  $x + 2y = 3z$ . By inspection we notice that for any values  $\beta \neq 0$  and  $\gamma \neq 0$ ,

$$\mathbf{v}_2 = \beta \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_3 = \gamma \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

are both in fact eigenvectors corresponding to  $\lambda = -3$ , i.e. there are two linearly independent eigenvectors corresponding to this (repeated) eigenvalue. (Note that  $A - \lambda I$  has rank 1 when  $\lambda = -3$ . This implies that the basis of solutions corresponding to  $\lambda = -3$  consists of two linearly independent vectors.)

*Algebraic and geometric multiplicity.* The degree of a root (eigenvalue) of the characteristic polynomial of a matrix—the number of times the root is repeated—is called the *algebraic multiplicity* of the eigenvalue.

The number of linearly independent eigenvectors corresponding to an eigenvalue is called the *geometric multiplicity* of the eigenvalue. It can be shown that for any eigenvalue

$$\text{geometric multiplicity} \leq \text{algebraic multiplicity}.$$

**7.1.8. Example.** Go back and examine this property for all of the eigenvalue problem examples above.

## 7.2. Practical application: mass–spring systems

**7.2.1. Example.** Two identical simple pendula oscillate in the plane as shown in Figure 7.1. Both pendula consist of light rods of length  $\ell = 10$  and are suspended from the same ceiling a distance  $L = 15$  apart, with equal masses  $m = 1$  attached to their ends. The angles the pendula make to the downward vertical are  $\theta_1$  and  $\theta_2$ , and they are coupled through the spring shown which has stiffness coefficient  $k = 1$ . The spring has unstretched length  $L = 15$ . You may also assume that the acceleration due to gravity  $g = 10$ .

- (a) Assuming that the oscillations of the spring remain small in amplitude, so that  $|\theta_1| \ll 1$  and  $|\theta_2| \ll 1$ , by applying Newton's second law and Hooke's law, show that the coupled pendula system gives rise to the system of differential equations

$$\frac{d^2\Theta}{dt^2} = A\Theta, \quad \text{where} \quad A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \quad (7.8)$$

and

$$\Theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

is the vector of unknown angles for each of the pendula shown in Figure 7.1.

- (b) By looking for a solution of the form  $\Theta(t) = \text{Re}\{Ce^{i\omega t}\}$  for a constant vector  $C$ , show that solving the system of differential equations (7.8) reduces to solving the eigenvalue problem

$$(A + \omega^2 I)C = \mathbf{0}. \quad (7.9)$$

- (c) Solve the eigenvalue problem (7.9) in part (b) above, stating clearly the eigenvalues and associated eigenvectors.
- (d) Hence enumerate the possible modes of oscillation of the masses corresponding to each eigenvalue-eigenvector pair.
- (e) Finally, write down the general solution of the system of equations (7.8).

## 7.3. Diagonalization

**7.3.1. Similarity transformations.** An  $n \times n$  matrix  $\tilde{A}$  is said to be *similar* to and  $n \times n$  matrix  $A$  if

$$\tilde{A} = X^{-1}AX$$

for some (non-singular)  $n \times n$  matrix  $X$ . This transformation, which gives  $\tilde{A}$  from  $A$ , is called a *similarity transformation*.

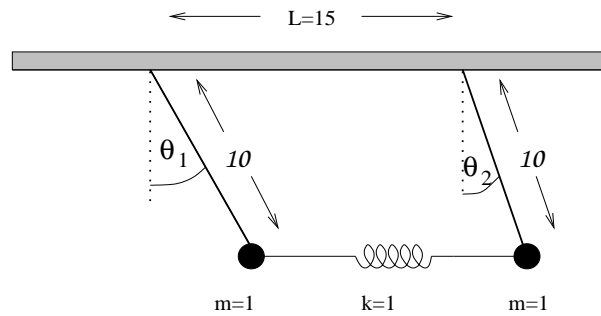


FIGURE 7.1. Simple coupled pendula system.

**7.3.2. Diagonalization transformation.** If an  $n \times n$  matrix  $A$  has a set of  $n$  linearly independent eigenvectors,  $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(m)}$ , then if we set  $X = [\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(m)}]$  (i.e. the matrix whose columns are the eigenvectors of  $A$ ), we have

$$X^{-1}AX = \Lambda,$$

where  $\Lambda$  is the  $n \times n$  diagonal matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix},$$

i.e. the matrix whose diagonal entries are the eigenvalues of the matrix  $A$  and whose all other entries are zero.

**7.3.3. Symmetric Matrices.** A *real* square matrix  $A$  is said to be *symmetric* if transposition leaves it unchanged, i.e.  $A^T = A$ . If  $A$  is a real-symmetric  $n \times n$  matrix, then

- its eigenvalues,  $\lambda_1, \dots, \lambda_n$ , are all real;
- the corresponding eigenvectors  $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}$  are linearly independent.

Hence in particular, all real-symmetric matrices are diagonalizable.

**7.3.4. Example.** Find the matrix  $X$  that diagonalizes the matrix

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

via a similarity transformation.

**7.3.5. Solution.** The eigenvalues of  $A$  are 1, 3 and 6. The corresponding eigenvectors are

$$\mathbf{x}^{(1)} = (-2, 1, 0)^T, \quad \mathbf{x}^{(2)} = (0, 0, 1)^T, \quad \mathbf{x}^{(3)} = (1, 2, 0)^T.$$

Hence

$$X = \begin{pmatrix} -2 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$

Now we can check: since

$$X^{-1} = \begin{pmatrix} -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 \\ \frac{1}{5} & \frac{2}{5} & 0 \end{pmatrix},$$

then

$$X^{-1}AX = \begin{pmatrix} -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 \\ \frac{1}{5} & \frac{2}{5} & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -2 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

## Systems of differential equations

### 8.1. Linear second order ODEs

Consider the linear second order constant coefficient homogeneous ODE:

$$au'' + bu' + cu = 0.$$

If we set  $v = u'$ , we can re-write this linear second order ODE as the following coupled system of linear first order ODEs (assuming  $a \neq 0$ ),

$$\begin{aligned} u' &= v, \\ v' &= -(c/a)u - (b/a)v, \end{aligned}$$

in the form

$$U' = AU,$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}.$$

To solve this system of linear constant coefficient first order ODEs, we look for a solution of the form

$$U = Ce^{\lambda t} \quad \text{where} \quad C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \text{constant vector.}$$

Substituting this ansatz<sup>1</sup> for the solution into the differential equation for  $U$  above, we soon see that indeed, this exponential form is a solution, provided we can find a  $\lambda$  for which there exists a non-trivial solution to

$$AC = \lambda C,$$

(after dividing through by the exponential factor  $e^{\lambda t}$ ).

To find non-trivial solutions  $C$  to this eigenvalue problem, we first solve the characteristic equation  $\det(A - \lambda I)C = 0$ , i.e.

$$\det \begin{pmatrix} -\lambda & 1 \\ -c/a & -b/a - \lambda \end{pmatrix} = 0 \quad \Leftrightarrow \quad a\lambda^2 + b\lambda + c = 0.$$

Suppose there are two solutions (eigenvalues), say  $\lambda_1$  and  $\lambda_2$ —in general real or complex. The corresponding eigenvectors satisfy

$$\begin{pmatrix} -\lambda & 1 \\ -c/a & -b/a - \lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{0},$$

or alternatively the system of equations

$$\begin{aligned} -\lambda c_1 + c_2 &= 0, \\ -(c/a)c_1 - (b/a + \lambda)c_2 &= 0. \end{aligned}$$

The first equation gives

$$c_2 = \lambda c_1,$$

---

<sup>1</sup>Ansatz is German for “approach”.

and we expect the second equation to give the same (which it does):

$$c_2 = -\frac{c/a}{\lambda + b/a} c_1 = -\frac{c\lambda}{a\lambda^2 + b\lambda} c_1 = \lambda c_1.$$

So for  $\lambda = \lambda_1$ , the corresponding eigenvector is

$$C^{(1)} = \alpha \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix},$$

for arbitrary values of  $\alpha \neq 0$ , whereas for  $\lambda = \lambda_2$ , the corresponding eigenvector is

$$C^{(2)} = \beta \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix},$$

for arbitrary values of  $\beta \neq 0$ .

Hence using the Principle of Superposition, we see that the final solution to our original second order ODE can be expressed in the form

$$U = C^{(1)}e^{\lambda_1 t} + C^{(2)}e^{\lambda_2 t} \equiv \alpha \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} e^{\lambda_1 t} + \beta \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} e^{\lambda_2 t}.$$

How does this solution compare with that given in Table 2.1?

## 8.2. Higher order linear ODEs

Recall from Chapter 1 that we can re-express any scalar  $n^{\text{th}}$  order ODE (linear or nonlinear) as a system of  $n$  first order ODEs. In particular, any  $n^{\text{th}}$  order linear ODE of the form (we will restrict ourselves to the homogeneous case for the moment)

$$a_n(t) \frac{d^n u}{dt^n} + a_{n-1}(t) \frac{d^{n-1} u}{dt^{n-1}} + \cdots + a_2(t) \frac{d^2 u}{dt^2} + a_1(t) \frac{du}{dt} + a_0(t)u = 0, \quad (8.1)$$

after we identify new variables  $u_1 = \frac{du}{dt}$ ,  $u_2 = \frac{du_1}{dt}$ ,  $\dots$ ,  $u_{n-1} = \frac{du_{n-2}}{dt}$ , is equivalent to the linear system of  $n$  first order ODEs (also set  $u_0 \equiv u$ )

$$\begin{aligned} \frac{du_0}{dt} &= u_1, \\ \frac{du_1}{dt} &= u_2, \\ &\vdots \\ \frac{du_{n-2}}{dt} &= u_{n-1}, \\ a_n(t) \frac{du_{n-1}}{dt} &= -a_{n-1}(t)u_{n-1} - \cdots - a_2(t)u_2 - a_1(t)u_1 - a_0(t)u_0. \end{aligned}$$

We can express this system of  $n$  first order linear ODEs much more succinctly in matrix form as

$$U' = A(t)U, \quad (8.2)$$

where  $U$  is the vector of unknowns  $u_0, \dots, u_{n-1}$  and

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \cdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \cdots & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-1}}{a_n} \end{pmatrix}.$$

### 8.3. Solution to linear constant coefficient ODE systems

**8.3.1. Solution in exponential form.** We shall also restrict ourselves to the case when the coefficient matrix  $A$  is constant. In the last section we saw how the coefficient matrix could look when we re-write a scalar  $n^{\text{th}}$  order ODE as a system of  $n$  first order ODEs (if we assumed in that case that the coefficients  $a_0, \dots, a_n$  were all constant, then the coefficient matrix  $A$  would also constant). Here we're interested in the case when  $A$  could be a constant matrix of any form (even with complex entries). Hence in general, our system of linear constant coefficient ODEs has the form

$$U' = AU. \quad (8.3)$$

If we are given the initial condition  $U(0) = U_0$ , then the solution to (8.3) is

$$U = \exp(At)U_0, \quad (8.4)$$

where we define the exponential function of a matrix, say  $B$ , as the sum of the matrix power series

$$\exp(B) = I + B + \frac{1}{2!}B^2 + \frac{1}{3!}B^3 + \dots. \quad (8.5)$$

We can quite quickly check that ' $\exp(At)U_0$ ' is indeed the solution to the initial value problem associated with (8.3), note that

$$\begin{aligned} \frac{d}{dt}(\exp(At)U_0) &= \frac{d}{dt}\left(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots\right)U_0 \\ &= \left(A + A^2t + \frac{1}{2!}A^3t^2 + \dots\right)U_0 \\ &= A\left(I + At + \frac{1}{2!}A^2t^2 + \dots\right)U_0 \\ &= A\exp(At)U_0. \end{aligned}$$

**8.3.2. Solution via diagonalization.** Suppose that the  $n \times n$  constant coefficient matrix  $A$  has eigenvalues  $\lambda_1, \dots, \lambda_n$  and  $n$  linearly independent eigenvectors. Set  $X$  to be the matrix whose  $n$  columns are the eigenvectors of  $A$ . For the initial value problem associated with (8.3), consider the linear transformation

$$U(t) = XV(t).$$

Differentiating this formula, we see that

$$U'(t) = XV'(t),$$

noting  $X$  is constant because  $A$  is. Substituting these last two formulae into (8.3) we see that we get (here  $\Lambda$  denotes the diagonal matrix with the eigenvalues of  $A$  down the diagonal)

$$\begin{aligned} &XV' = AXV \\ \Leftrightarrow &V' = \underbrace{X^{-1}AX}_{\equiv \Lambda} V \\ \Leftrightarrow &V' = \Lambda V \\ \Leftrightarrow &v'_i = \lambda_i v_i \quad \text{for each } i = 1, \dots, n \\ \Leftrightarrow &v_i(t) = e^{\lambda_i t} v_i(0) \\ \Leftrightarrow &V(t) = \exp(\Lambda t)V(0) \\ \Leftrightarrow &XU(t) = \exp(\Lambda t)XU_0 \\ \Leftrightarrow &U(t) = X^{-1} \exp(\Lambda t)XU_0. \end{aligned}$$

Hence the solution to (8.3) can also be expressed in the form

$$U = X^{-1} \exp(\Lambda t) X U_0.$$

Note that, using the uniqueness of solutions to linear ODEs, we have also established that

$$\exp(At) \equiv X^{-1} \exp(\Lambda t) X.$$

#### 8.4. Solution to general linear ODE systems

When the coefficient matrix is not constant, then the solution to the initial value problem

$$U' = A(t)U, \quad U(0) = U_0, \quad (8.6)$$

is given by the Neumann series

$$U(t) = \left( I + \int_0^t A(\tau) d\tau + \int_0^t A(\tau_1) \int_0^{\tau_1} A(\tau_2) d\tau_2 d\tau_1 + \dots \right) U_0.$$

Check that it is indeed correct by differentiating the series term by term and showing that it satisfies the linear ODE system (8.6). This Neumann series converges provided  $\int_0^t \|A(\tau)\| d\tau < \infty$ .

## APPENDIX A

### A.1. Euler's formula

For any real angle  $\theta$  measured in radians,

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

To prove this we use the series expansion definition for  $e^z$ :

$$\begin{aligned} e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \end{aligned}$$

This series is known to converge for all real and complex  $z$ . Let  $z = i\theta$ , with  $\theta$  real, and noting that  $i^2 = -1$ ,  $i^3 = -i$ ,  $i^4 = 1$ ,  $i^5 = i$ , ... etc. , then for every real  $\theta$ :

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \cdots \\ &= \left( 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots \right) + i \left( \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots \right) \\ &= \cos \theta + i \sin \theta. \end{aligned}$$

We have used that the Taylor series expansion for a given function  $f(\theta)$  about  $\theta = a$  is unique (in this case  $\cos \theta$  and  $\sin \theta$  about  $\theta = 0$ ).

## A.2. Electrical circuits and networks

**A.2.1. Simple electrical circuit.** Consider the flow of an electrical current  $I(t)$  in a simple series circuit as shown in Figure A.1. The resistor has resistance  $R$  *Ohms*, the capacitor has capacitance  $C$  *Farads* and the inductor has inductance  $\ell$  *Henrys* which are positive constants. A battery or power source provides an impressed voltage of  $V(t)$  *Volts* at any given time. We have the following relations:

- The rate of change of total charge  $Q(t)$  *Coulombs* in the capacitor at time  $t$ , is the current,

$$I(t) = \frac{dQ}{dt}.$$

- The voltage drop across the capacitor is  $Q/C$ .
- The voltage drop across the inductor is  $\ell \frac{dI}{dt}$ .
- *Ohm's law*: The voltage drop  $V$  across any resistor of resistance  $R$  Ohms is given by  $V = IR$ , where  $I$  (measured in Amps) is the current passing through the resistor.
- *Kirchhoff's Voltage law*: in a closed circuit, the impressed voltage is equal to the sum of the voltage drops in the rest of the circuit.

Combining all these relations, we see that for the circuit in Figure A.1,

$$\ell \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C}Q = V(t). \quad (\text{A.1})$$

'Feedback-squeals' in electric circuits at concerts are an example of resonance effects in such equations. Note also that, from Kirchhoff's law, there is a nontrivial relationship between the initial values:  $I'(0) = (V_0 - RI_0 - Q_0/C)/\ell$ .

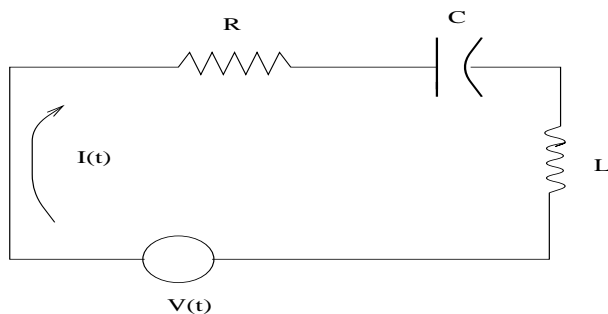


FIGURE A.1. Simple electrical circuit.

**A.2.2. Electrical networks.** Consider the following laws for electrical circuits.

- *Kirchhoff's node rule:* At any point of a circuit (including the battery), the sum of the inflowing currents equals the sum of the outgoing currents.
- *Kirchhoff's loop rule:* In any closed loop, the sum of the voltage drops equals the impressed electromotive force.

Combining these laws together with Ohm's Law from section A.2.1, we can write down the system of linear algebraic equations for the currents in various parts of the circuit shown in the Figure A.2:

$$\begin{array}{rcl}
 \text{Node P:} & I_1 - I_2 + I_3 & = 0, \\
 \text{Node Q:} & -I_1 + I_2 - I_3 & = 0, \\
 \text{Right loop:} & 10I_2 + 25I_3 & = 90, \\
 \text{Left loop:} & 20I_1 + 10I_2 & = 80.
 \end{array} \tag{A.2}$$

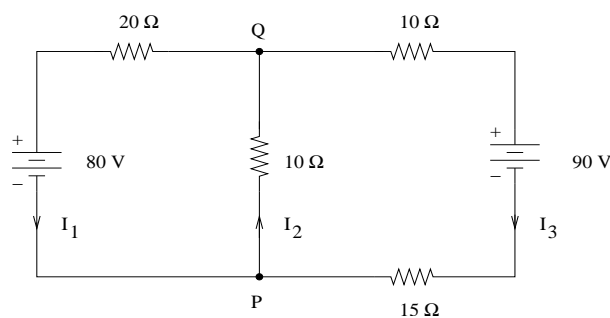


FIGURE A.2. Simple electrical network.

### A.3. Determinants and inverse matrices

**A.3.1. Identity matrix.** A *square matrix* is one in which the number of rows and columns are equal. An *identity matrix* is a square matrix whose elements are all zero except those on the *leading diagonal* which are unity ( $= 1$ ). The leading diagonal is the one from the top left to the bottom right. The identity matrix is denoted by  $I$  (or  $I_n$  to show that it has order  $n \times n$ ). For example

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The identity matrix has the property that  $AI = IA = A$  for any matrix  $A$ .

**A.3.2. Inverse matrices.** If  $A$  is a square matrix, then provided it exists, the inverse matrix of  $A$ , denoted  $A^{-1}$ , has the property that

$$A^{-1}A = AA^{-1} = I.$$

It is important to note that not every square matrix has an inverse. If we can find  $A^{-1}$  then we can solve the system  $A\mathbf{x} = \mathbf{b}$  because

$$A\mathbf{x} = \mathbf{b} \Rightarrow A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \Rightarrow I\mathbf{x} = A^{-1}\mathbf{b} \Rightarrow \mathbf{x} = A^{-1}\mathbf{b}.$$

Thus the solution is simply  $\mathbf{x} = A^{-1}\mathbf{b}$ .

**A.3.3. Determinants.** Recall our discussion of  $2 \times 2$  systems. We found that the determinant  $D = a_{11}a_{22} - a_{12}a_{21}$  was significant. We found that there was a unique solution provided  $D \neq 0$ . If  $D \neq 0$  the matrix  $A$  has an inverse  $A^{-1}$  given by

$$A^{-1} = \frac{1}{D} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

If  $D = 0$ , the matrix  $A$  has no inverse.

#### A.3.4. Example.

$$A = \begin{pmatrix} -1 & 5 \\ 2 & 3 \end{pmatrix} \Rightarrow \det A = -13 \neq 0 \Rightarrow A^{-1} = -\frac{1}{13} \begin{pmatrix} 3 & -5 \\ -2 & -1 \end{pmatrix}.$$

We can check that this is indeed the inverse by testing for the properties of an inverse matrix:

$$\begin{aligned} AA^{-1} &= -\frac{1}{13} \begin{pmatrix} -1 & 5 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & -5 \\ -2 & -1 \end{pmatrix} \\ &= -\frac{1}{13} \begin{pmatrix} -3 - 10 & 5 - 5 \\ 6 - 6 & -10 - 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

These ideas can be extended from  $2 \times 2$  matrices to  $3 \times 3$ ,  $\dots$ ,  $n \times n$  etc. Again the determinant will dictate whether the matrix is *singular* (has no inverse) or *nonsingular* (has an inverse). Given a  $3 \times 3$  matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

we define its determinant to be

$$\det A = a_{11} \cdot \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \cdot \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \cdot \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}.$$

**A.3.5. Example.**

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{pmatrix} = 1 \cdot \det \begin{pmatrix} 3 & 5 \\ 5 & 12 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 1 & 5 \\ 1 & 12 \end{pmatrix} + 3 \cdot \det \begin{pmatrix} 1 & 3 \\ 1 & 5 \end{pmatrix} = 3.$$

The value of  $\det A$  may be obtained by expanding about *any* row or column (not just the first row as we have done) using the *chessboard* pattern of signs and the “cover up” method.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

**A.3.6. Example.** Let  $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{pmatrix}$ . Here are three of the possible ways to calculate

$\det A$ :

- Expand about the first row to give

$$\det A = +1 \det \begin{pmatrix} 6 & 4 \\ 0 & 2 \end{pmatrix} - 3 \det \begin{pmatrix} 2 & 4 \\ -1 & 2 \end{pmatrix} + 0 \det \begin{pmatrix} 2 & 6 \\ -1 & 0 \end{pmatrix} = -12.$$

- Expand about the middle row to give

$$\det A = -2 \det \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} + 6 \det \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} - 4 \det \begin{pmatrix} 1 & 3 \\ -1 & 0 \end{pmatrix} = -12.$$

- Expand about the last column to give

$$\det A = 0 \det \begin{pmatrix} 2 & 6 \\ -1 & 0 \end{pmatrix} - 4 \det \begin{pmatrix} 1 & 3 \\ -1 & 0 \end{pmatrix} + 2 \det \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} = -12.$$

**A.3.7. Matrix transpose.** The *transpose*  $A^T$  of the  $m \times n$  matrix  $A$  is the  $n \times m$  matrix obtained by interchanging rows and columns.

**A.3.8. Example.** If

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

then their transposes are

$$A^T = \begin{pmatrix} 3 & 4 \\ 2 & 5 \\ 1 & 6 \end{pmatrix} \quad \text{and} \quad B^T = (1 \ 4).$$



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